

Finite dimensional simple modules of deformed current Lie algebras

Kentaro Wada

ABSTRACT. The deformed current Lie algebra was introduced in [W] to study the representation theory of cyclotomic q -Schur algebras at $q = 1$. In this paper, we classify finite dimensional simple modules of deformed current Lie algebras.

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§ 0. INTRODUCTION

0.1. The deformed current Lie algebra $\mathfrak{g}_{\hat{\mathbf{Q}}}(\mathbf{m})$ was introduced in [W] to study the representation theory of cyclotomic q -Schur algebras at $q = 1$. In this paper, we introduce the deformed current Lie algebra $\mathfrak{sl}_m^{(\mathbf{Q})}[x]$ and $\mathfrak{gl}_m^{(\mathbf{Q})}[x]$ over \mathbb{C} associated with the special linear Lie algebra \mathfrak{sl}_m and general linear Lie algebra \mathfrak{gl}_m respectively. $\mathfrak{sl}_m^{(\mathbf{Q})}[x]$ (resp. $\mathfrak{gl}_m^{(\mathbf{Q})}[x]$) is a deformation of the current Lie algebra $\mathfrak{sl}_m[x] = \mathfrak{sl}_m \otimes_{\mathbb{C}} \mathbb{C}[x]$ (resp. $\mathfrak{gl}_m[x] = \mathfrak{gl}_m \otimes_{\mathbb{C}} \mathbb{C}[x]$) with deformation parameters $\mathbf{Q} = (Q_1, Q_2, \dots, Q_{m-1}) \in \mathbb{C}^{m-1}$. Note that $\mathfrak{sl}_m^{(\mathbf{Q})}[x]$ (resp. $\mathfrak{gl}_m^{(\mathbf{Q})}[x]$) is coincide with $\mathfrak{sl}_m[x]$ (resp. $\mathfrak{gl}_m[x]$) if $Q_i = 0$ for all $i = 1, 2, \dots, m-1$. The Lie algebra $\mathfrak{g}_{\hat{\mathbf{Q}}}(\mathbf{m})$ introduced in [W] is isomorphic to $\mathfrak{gl}_m^{(\mathbf{Q})}[x]$ under a suitable choice of deformation parameters \mathbf{Q} (Lemma 1.7).

0.2. The differences of the representation theory of $\mathfrak{sl}_m^{(\mathbf{Q})}[x]$ from one of $\mathfrak{sl}_m[x]$ appear in the following two points. The deformed current Lie algebra $\mathfrak{sl}_m^{(\mathbf{Q})}[x]$ has a family

of 1-dimensional representations $\{\mathcal{L}^\beta \mid \beta \in \prod_{i=1}^{m-1} \mathbb{B}^{\langle Q_i \rangle}\}$, where

$$\mathbb{B}^{\langle Q_i \rangle} = \begin{cases} \{0\} & \text{if } Q_i = 0, \\ \mathbb{C} & \text{if } Q_i \neq 0, \end{cases}$$

although the 1-dimensional representation of $\mathfrak{sl}_m[x]$ is only the trivial representation (Lemma 6.2). (We remark that $\mathcal{L}^{(0,\dots,0)}$ is the trivial representation of $\mathfrak{sl}_m^{(Q)}[x]$.)

The second difference appears in the evaluation modules. For each $\gamma \in \mathbb{C}$, we can consider the evaluation homomorphism $\mathbf{ev}_\gamma : U(\mathfrak{sl}_m^{(Q)}[x]) \rightarrow U(\mathfrak{sl}_m)$ which is a deformation of the evaluation homomorphism for $\mathfrak{sl}_m[x]$ (see the paragraph 1.5 for the definition). Then we can consider the evaluation modules by regarding $U(\mathfrak{sl}_m)$ -modules as $U(\mathfrak{sl}_m^{(Q)}[x])$ -modules through the evaluation homomorphism \mathbf{ev}_γ . The evaluation homomorphism \mathbf{ev}_γ is surjective if $\gamma \neq Q_i^{-1}$ for all $i = 1, 2, \dots, m-1$ such that $Q_i \neq 0$. However, \mathbf{ev}_γ is not surjective if $\gamma = Q_i^{-1}$ for some $i = 1, 2, \dots, m-1$. Moreover, in general, the evaluation module of a simple $U(\mathfrak{sl}_m)$ -module at $\gamma \in \mathbb{C}$ is not simple if $\gamma = Q_i^{-1}$ for some $i = 1, 2, \dots, m-1$ (see Remark 5.10).

0.3. It is a purpose of this paper to classify the finite dimensional simple modules of $\mathfrak{sl}_m^{(Q)}[x]$ and $\mathfrak{gl}_m^{(Q)}[x]$. A classification of the finite dimensional simple modules for the original current Lie algebra is well-known (e.g. [C], [CP]). The classification for $\mathfrak{sl}_m^{(Q)}[x]$ (resp. $\mathfrak{gl}_m^{(Q)}[x]$) is an analogue of the original case.

Since $\mathfrak{sl}_m^{(Q)}[x]$ has the triangular decomposition (Proposition 1.4), we can develop the usual highest weight theory (see §2). In particular, any finite dimensional simple $U(\mathfrak{sl}_m^{(Q)}[x])$ -module is isomorphic to a highest weight module $\mathcal{L}(\mathbf{u})$ of highest weight $\mathbf{u} \in \prod_{i=1}^{m-1} \prod_{t \geq 0} \mathbb{C}$ (Proposition 2.6). Then it is enough to determine the highest weights such that the corresponding simple highest weight modules are finite dimensional. We obtain a classification of such highest weights as follows. Let $\mathbb{C}[x]_{\text{monic}}$ be the set of monic polynomials over \mathbb{C} with the indeterminate variable x . For each $Q \in \mathbb{C}$, put

$$\mathbb{C}[x]_{\text{monic}}^{\langle Q \rangle} = \begin{cases} \mathbb{C}[x]_{\text{monic}} & \text{if } Q = 0, \\ \{\varphi \in \mathbb{C}[x]_{\text{monic}} \mid Q^{-1} \text{ is not a root of } \varphi\} & \text{if } Q \neq 0. \end{cases}$$

We define the map $\prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^{\langle Q_i \rangle} \times \mathbb{B}^{\langle Q_i \rangle}) \rightarrow \prod_{i=1}^{m-1} \prod_{t \geq 0} \mathbb{C}$,

$$(\varphi, \beta) = ((\varphi_i, \beta_i))_{1 \leq i \leq m-1} \mapsto \mathbf{u}^{(Q)}(\varphi, \beta) = (\mathbf{u}^{(Q)}(\varphi, \beta)_{i,t})_{1 \leq i \leq m-1, t \geq 0},$$

by

$$(0.3.1) \quad \mathbf{u}^{(Q)}(\varphi, \beta)_{i,t} = \begin{cases} \gamma_{i,1}^t + \gamma_{i,2}^t + \cdots + \gamma_{i,n_i}^t & \text{if } Q_i = 0, \\ \gamma_{i,1}^t + \gamma_{i,2}^t + \cdots + \gamma_{i,n_i}^t + Q_i^{-t} \beta_i & \text{if } Q_i \neq 0 \end{cases}$$

when $\varphi_i = (x - \gamma_{i,1})(x - \gamma_{i,2}) \dots (x - \gamma_{i,n_i})$ ($1 \leq i \leq m-1$). Then we have the following classification of finite dimensional simple $U(\mathfrak{sl}_m^{(Q)}[x])$ -modules (Theorem 6.4).

Theorem: $\{\mathcal{L}(\mathbf{u}^{(Q)}(\varphi, \beta)) \mid (\varphi, \beta) \in \prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^{(Q_i)} \times \mathbb{B}^{(Q_i)})\}$ gives a complete set of isomorphism classes of finite dimensional simple $U(\mathfrak{sl}_m^{(Q)}[x])$ -modules.

We remark that $\mathcal{L}(\mathbf{u}^{(Q)}(\varphi, \beta))$ is isomorphic to a subquotient of

$$\left(\bigotimes_{j=1}^{m-1} \bigotimes_{k=1}^{n_j} L(\omega_j)^{\mathbf{ev}_{\gamma_j, k}} \right) \otimes \mathcal{L}^\beta,$$

where $\{\omega_j \mid 1 \leq j \leq m-1\}$ is the set of fundamental weights for \mathfrak{sl}_m , $L(\omega_j)$ ($1 \leq j \leq m-1$) is the simple highest weight $U(\mathfrak{sl}_m)$ -module of highest weight ω_j and $L(\omega_j)^{\mathbf{ev}_{\gamma_j, k}}$ is the evaluation module of $L(\omega_j)$ at $\gamma_{j,k}$.

We also see that any finite dimensional simple $U(\mathfrak{gl}_m^{(Q)}[x])$ -module is isomorphic to a highest weight module $\mathcal{L}(\tilde{\mathbf{u}})$ of highest weight $\tilde{\mathbf{u}} \in \prod_{j=1}^m \prod_{t \geq 0} \mathbb{C}$ (Proposition 3.3). Note that $\mathfrak{sl}_m^{(Q)}[x]$ is a Lie subalgebra of $\mathfrak{gl}_m^{(Q)}[x]$ (Proposition 1.4 (iii)). The difference of representations of $\mathfrak{gl}_m^{(Q)}[x]$ from one of $\mathfrak{sl}_m^{(Q)}[x]$ is given by the family of 1-dimensional $U(\mathfrak{gl}_m^{(Q)}[x])$ -modules $\{\tilde{\mathcal{L}}^{\mathbf{h}} \mid \mathbf{h} \in \prod_{t \geq 0} \mathbb{C}\}$. We remark that $\tilde{\mathcal{L}}^{\mathbf{h}}$ ($\mathbf{h} \in \prod_{t \geq 0} \mathbb{C}$) is isomorphic to the trivial representation $\mathcal{L}^{(0, \dots, 0)}$ as a $U(\mathfrak{sl}_m^{(Q)}[x])$ -module when we restrict the action. We obtain the classification of finite dimensional simple $U(\mathfrak{gl}_m^{(Q)}[x])$ -modules as follows. We define the map $\prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^{(Q_i)} \times \mathbb{B}^{(Q_i)}) \times \prod_{t \geq 0} \mathbb{C} \rightarrow \prod_{j=1}^m \prod_{t \geq 0} \mathbb{C}$,

$$(\varphi, \beta, \mathbf{h}) = ((\varphi_i, \beta_i)_{1 \leq i \leq m-1}, (h_t)_{t \geq 0}) \mapsto \tilde{\mathbf{u}}^{(Q)}(\varphi, \beta, \mathbf{h}) = (\tilde{\mathbf{u}}^{(Q)}(\varphi, \beta, \mathbf{h})_{j,t})_{1 \leq j \leq m, t \geq 0}$$

by

$$\tilde{\mathbf{u}}^{(Q)}(\varphi, \beta, \mathbf{h})_{j,t} = \begin{cases} \sum_{k=j}^{m-1} \mathbf{u}^{(Q)}(\varphi, \beta)_{k,t} + h_t & \text{if } 1 \leq j \leq m-1 \text{ and } t \geq 0, \\ h_t & \text{if } j = m \text{ and } t \geq 0, \end{cases}$$

where $\mathbf{u}^{(Q)}(\varphi, \beta)_{k,t}$ is determined by (0.3.1). Then we have the following classification of finite dimensional simple $U(\mathfrak{gl}_m^{(Q)}[x])$ -modules (Theorem 7.4).

Theorem: $\{\mathcal{L}(\tilde{\mathbf{u}}^{(Q)}(\varphi, \beta, \mathbf{h})) \mid (\varphi, \beta, \mathbf{h}) \in \prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^{(Q_i)} \times \mathbb{B}^{(Q_i)}) \times \prod_{t \geq 0} \mathbb{C}\}$ gives a complete set of isomorphism classes of finite dimensional simple $U(\mathfrak{gl}_m^{(Q)}[x])$ -modules.

We remark that $\mathcal{L}(\tilde{\mathbf{u}}^{(Q)}(\varphi, \beta, \mathbf{h}))$ is isomorphic to a subquotient of

$$\left(\bigotimes_{j=1}^{m-1} \bigotimes_{k=1}^{n_j} L(\tilde{\omega}_j)^{\tilde{\mathbf{ev}}_{\gamma_j, k}} \right) \otimes \tilde{\mathcal{L}}^\beta \otimes \tilde{\mathcal{L}}^{\mathbf{h}}.$$

(See §7 for definitions of $L(\tilde{\omega}_j)^{\widetilde{\text{ev}}_{\gamma_j, k}}$, $\tilde{\mathcal{L}}^\beta$ and $\tilde{\mathcal{L}}^{\text{h}}$.) We also remark that

$$\begin{aligned} \mathcal{L}(\tilde{\mathbf{u}}^{(\mathbf{Q})}(\varphi, \beta, \mathbf{h})) &\cong \mathcal{L}(\mathbf{u}^{(\mathbf{Q})}(\varphi, \beta)), \\ L(\tilde{\omega}_j)^{\widetilde{\text{ev}}_{\gamma_j, k}} &\cong L(\omega_j)^{\text{ev}_{\gamma_j, k}}, \quad \tilde{\mathcal{L}}^\beta \cong \mathcal{L}^\beta \text{ and } \tilde{\mathcal{L}}^{\text{h}} \cong \mathcal{L}^{(0, \dots, 0)} \end{aligned}$$

as $U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])$ -modules when we restrict the action.

Acknowledgements: This work was supported by JSPS KAKENHI Grant Number JP16K17565.

§ 1. DEFORMED CURRENT LIE ALGEBRAS $\mathfrak{sl}_m^{(\mathbf{Q})}[x]$ AND $\mathfrak{gl}_m^{(\mathbf{Q})}[x]$

In this section, we give a definition of deformed current Lie algebras $\mathfrak{sl}_m^{(\mathbf{Q})}[x]$ and $\mathfrak{gl}_m^{(\mathbf{Q})}[x]$, and also give some basic facts. The definition of $\mathfrak{gl}_m^{(\mathbf{Q})}[x]$ in this section is different from one of $\mathfrak{g}_{\hat{\mathbf{Q}}}(\mathbf{m})$ given in [W]. The relation between $\mathfrak{gl}_m^{(\mathbf{Q})}[x]$ and $\mathfrak{g}_{\hat{\mathbf{Q}}}(\mathbf{m})$ is given in Lemma 1.7.

Definition 1.1. Put $\mathbf{Q} = (Q_1, Q_2, \dots, Q_{m-1}) \in \mathbb{C}^{m-1}$. We define the Lie algebra $\mathfrak{sl}_m^{(\mathbf{Q})}[x]$ over \mathbb{C} by the following generators and defining relations:

Generators: $\mathcal{X}_{i,t}^\pm, \mathcal{J}_{i,t}$ ($1 \leq i \leq m-1, t \geq 0$).

Relations:

- (L1) $[\mathcal{J}_{i,s}, \mathcal{J}_{j,t}] = 0,$
- (L2) $[\mathcal{J}_{j,s}, \mathcal{X}_{i,t}^\pm] = \pm a_{ji} \mathcal{X}_{i,s+t}^\pm,$
- (L3) $[\mathcal{X}_{i,t}^+, \mathcal{X}_{j,s}^-] = \delta_{ij}(\mathcal{J}_{i,s+t} - Q_i \mathcal{J}_{i,s+t+1}),$
- (L4) $[\mathcal{X}_{i,t}^\pm, \mathcal{X}_{j,s}^\pm] = 0 \quad \text{if } j \neq i \pm 1,$
- (L5) $[\mathcal{X}_{i,t+1}^+, \mathcal{X}_{i\pm 1,s}^+] = [\mathcal{X}_{i,t}^+, \mathcal{X}_{i\pm 1,s+1}^+], \quad [\mathcal{X}_{i,t+1}^-, \mathcal{X}_{i\pm 1,s}^-] = [\mathcal{X}_{i,t}^-, \mathcal{X}_{i\pm 1,s+1}^-],$
- (L6) $[\mathcal{X}_{i,s}^+, [\mathcal{X}_{i,t}^+, \mathcal{X}_{i\pm 1,u}^+]] = [\mathcal{X}_{i,s}^-, [\mathcal{X}_{i,t}^-, \mathcal{X}_{i\pm 1,u}^-]] = 0,$

$$\text{where we put } a_{ji} = \begin{cases} 2 & \text{if } j = i, \\ -1 & \text{if } j = i \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

We also define the Lie algebra $\mathfrak{gl}_m^{(\mathbf{Q})}[x]$ over \mathbb{C} by the following generators and defining relations:

Generators: $\mathcal{X}_{i,t}^\pm$ ($1 \leq i \leq m-1, t \geq 0$), $\mathcal{I}_{j,t}$ ($1 \leq j \leq m, t \geq 0$).

Relations:

- (L'1) $[\mathcal{I}_{i,s}, \mathcal{I}_{j,t}] = 0,$
- (L'2) $[\mathcal{I}_{j,s}, \mathcal{X}_{i,t}^\pm] = \pm a'_{ji} \mathcal{X}_{i,s+t}^\pm,$
- (L'3) $[\mathcal{X}_{i,t}^+, \mathcal{X}_{j,s}^-] = \delta_{ij}(\mathcal{J}_{i,s+t} - Q_i \mathcal{J}_{i,s+t+1}), \text{ where we put } \mathcal{J}_{i,t} = \mathcal{I}_{i,t} - \mathcal{I}_{i+1,t},$

together with the relations (L4)-(L6) in the above. In the relation (L'2), we

$$\text{put } a'_{ji} = \begin{cases} 1 & \text{if } j = i, \\ -1 & \text{if } j = i + 1, \\ 0 & \text{otherwise.} \end{cases}$$

1.2. We call $\mathfrak{sl}_m^{(\mathbf{Q})}[x]$ (resp. $\mathfrak{gl}_m^{(\mathbf{Q})}[x]$) the deformed current Lie algebra associated with the special linear Lie algebra \mathfrak{sl}_m (resp. the general linear Lie algebra \mathfrak{gl}_m). If $Q_i = 0$ for all $i = 1, 2, \dots, m-1$, then $\mathfrak{sl}_m^{(\mathbf{Q})}[x]$ (resp. $\mathfrak{gl}_m^{(\mathbf{Q})}[x]$) coincides with the current Lie algebra $\mathfrak{sl}_m[x] = \mathfrak{sl}_m \otimes_{\mathbb{C}} \mathbb{C}[x]$ (resp. $\mathfrak{gl}_m[x] = \mathfrak{gl}_m \otimes_{\mathbb{C}} \mathbb{C}[x]$) associated with \mathfrak{sl}_m (resp. \mathfrak{gl}_m). We can also regard $\mathfrak{sl}_m^{(\mathbf{Q})}[x]$ (resp. $\mathfrak{gl}_m^{(\mathbf{Q})}[x]$) as a filtered deformation of $\mathfrak{sl}_m[x]$ (resp. $\mathfrak{gl}_m[x]$) in a similar way as in [W, Proposition 2.13].

1.3. For $1 \leq i \neq j \leq m$ and $t \geq 0$, we define an element $\mathcal{E}_{i,j;t} \in \mathfrak{sl}_m^{(\mathbf{Q})}[x]$ (resp. $\mathcal{E}_{i,j;t} \in \mathfrak{gl}_m^{(\mathbf{Q})}[x]$) by

$$\mathcal{E}_{i,j;t} = \begin{cases} [\mathcal{X}_{i,0}^+, [\mathcal{X}_{i+1,0}^+, \dots, [\mathcal{X}_{j-2,0}^+, \mathcal{X}_{j-1,t}^+] \dots]] & \text{if } j > i, \\ [\mathcal{X}_{i-1,0}^-, [\mathcal{X}_{i-2,0}^-, \dots, [\mathcal{X}_{j+1,0}^-, \mathcal{X}_{j,t}^-] \dots]] & \text{if } j < i. \end{cases}$$

In particular, we have $\mathcal{E}_{i,i+1;t} = \mathcal{X}_{i,t}^+$ and $\mathcal{E}_{i+1,i;t} = \mathcal{X}_{i,t}^-$.

Let \mathbf{n}^+ and \mathbf{n}^- be the Lie subalgebra of $\mathfrak{sl}_m^{(\mathbf{Q})}[x]$ (also of $\mathfrak{gl}_m^{(\mathbf{Q})}[x]$) generated by

$$\{\mathcal{X}_{i,t}^+ \mid 1 \leq i \leq m-1, t \geq 0\} \text{ and } \{\mathcal{X}_{i,t}^- \mid 1 \leq i \leq m-1, t \geq 0\}$$

respectively. Let \mathbf{n}^0 (resp. $\tilde{\mathbf{n}}^0$) be the Lie subalgebra of $\mathfrak{sl}_m^{(\mathbf{Q})}[x]$ (resp. $\mathfrak{gl}_m^{(\mathbf{Q})}[x]$) generated by

$$\{\mathcal{J}_{i,t} \mid 1 \leq i \leq m-1, t \geq 0\} \quad (\text{resp. } \{\mathcal{I}_{i,t} \mid 1 \leq i \leq m, t \geq 0\}).$$

By the relation (L1) (resp. (L'1)), we see that \mathbf{n}^0 (resp. $\tilde{\mathbf{n}}^0$) is a commutative Lie subalgebra of $\mathfrak{sl}_m^{(\mathbf{Q})}[x]$ (resp. $\mathfrak{gl}_m^{(\mathbf{Q})}[x]$).

Proposition 1.4.

- (i) $\{\mathcal{E}_{i,j;t} \mid 1 \leq i \neq j \leq m, t \geq 0\} \cup \{\mathcal{J}_{i,t} \mid 1 \leq i \leq m-1, t \geq 0\}$ gives a basis of $\mathfrak{sl}_m^{(\mathbf{Q})}[x]$.
- (ii) $\{\mathcal{E}_{i,j;t} \mid 1 \leq i \neq j \leq m, t \geq 0\} \cup \{\mathcal{I}_{j,t} \mid 1 \leq j \leq m, t \geq 0\}$ gives a basis of $\mathfrak{gl}_m^{(\mathbf{Q})}[x]$.
- (iii) There exists an injective homomorphism of Lie algebras

$$\Upsilon : \mathfrak{sl}_m^{(\mathbf{Q})}[x] \rightarrow \mathfrak{gl}_m^{(\mathbf{Q})}[x] \text{ such that } \mathcal{X}_{i,t}^{\pm} \mapsto \mathcal{X}_{i,t}^{\pm}, \text{ and } \mathcal{J}_{i,t} \mapsto \mathcal{I}_{i,t} - \mathcal{I}_{i+1,t}.$$

- (iv) We have the triangular decomposition

$$\mathfrak{sl}_m^{(\mathbf{Q})}[x] = \mathbf{n}^- \oplus \mathbf{n}^0 \oplus \mathbf{n}^+ \text{ and } \mathfrak{gl}_m^{(\mathbf{Q})}[x] = \mathbf{n}^- \oplus \tilde{\mathbf{n}}^0 \oplus \mathbf{n}^+ \quad (\text{as vector spaces}).$$

In particular,

$$\{\mathcal{E}_{i,j;t} \mid 1 \leq i < j \leq m, t \geq 0\} \quad (\text{resp. } \{\mathcal{E}_{i,j;t} \mid 1 \leq j < i \leq m, t \geq 0\})$$

gives a basis of \mathfrak{n}^+ (resp. \mathfrak{n}^-), and

$$\{\mathcal{J}_{i,t} \mid 1 \leq i \leq m-1, t \geq 0\} \quad (\text{resp. } \{\mathcal{I}_{j,t} \mid 1 \leq j \leq m, t \geq 0\})$$

gives a basis of \mathfrak{n}^0 (resp. $\tilde{\mathfrak{n}}^0$).

Proof. (i) and (ii) are proven in a similar way as in the proof of [W, Proposition 2.6]. By checking the defining relations, we see that Υ is well-defined. We also see that Υ is injective by investigating the basis given in (i) and (ii) under the homomorphism Υ . Then we have (iii). (iv) folloes from (i) and (ii). \square

1.5. Evaluation homomorphisms and evaluation modules. The general linear Lie algebra \mathfrak{gl}_m is a Lie algebra over \mathbb{C} generated by e_i, f_i ($1 \leq i \leq m-1$) and K_j ($1 \leq j \leq m$) together with the following defining relations:

$$\begin{aligned} [K_i, K_j] &= 0, \quad [K_j, e_i] = a'_{ji}e_i, \quad [K_j, f_i] = -a'_{ji}f_i, \\ [e_i, f_j] &= \delta_{ij}H_i, \text{ where } H_i = K_i - K_{i+1}, \\ [e_i, e_j] &= [f_i, f_j] = 0 \text{ if } j \neq i \pm 1, \quad [e_i, [e_i, e_{i\pm 1}]] = [f_i, [f_i, f_{i\pm 1}]] = 0. \end{aligned}$$

The special linear Lie algebra \mathfrak{sl}_m is a Lie subalgebra of \mathfrak{gl}_m generated by e_i, f_i, H_i ($1 \leq i \leq m-1$).

For each $\gamma \in \mathbb{C}$, by checking the defining relations, we have the homomorphisms of algebras (evaluation homomorphism)

$$\mathbf{ev}_\gamma : U(\mathfrak{sl}_m^{(\mathbb{Q})}[x]) \rightarrow U(\mathfrak{sl}_m) \text{ by } \mathcal{X}_{i,t}^+ \mapsto (1 - Q_i\gamma)\gamma^t e_i, \mathcal{X}_{i,t}^- \mapsto \gamma^t f_i, \mathcal{J}_{i,t} \mapsto \gamma^t H_i$$

and

$$\widetilde{\mathbf{ev}}_\gamma : U(\mathfrak{gl}_m^{(\mathbb{Q})}[x]) \rightarrow U(\mathfrak{gl}_m) \text{ by } \mathcal{X}_{i,t}^+ \mapsto (1 - Q_i\gamma)\gamma^t e_i, \mathcal{X}_{i,t}^- \mapsto \gamma^t f_i, \mathcal{I}_{j,t} \mapsto \gamma^t K_j.$$

Clearly, the homomorphism \mathbf{ev}_γ (resp. $\widetilde{\mathbf{ev}}_\gamma$) is surjective if $\gamma \neq Q_i^{-1}$ for all $i = 1, \dots, m-1$ such that $Q_i \neq 0$.

For a $U(\mathfrak{sl}_m)$ -module M (resp. a $U(\mathfrak{gl}_m)$ -module M), we can regard M as a $U(\mathfrak{sl}_m^{(\mathbb{Q})}[x])$ -module (resp. a $U(\mathfrak{gl}_m^{(\mathbb{Q})}[x])$ -module) through the evaluation homomorphism \mathbf{ev}_γ (resp. $\widetilde{\mathbf{ev}}_\gamma$). We call it the evaluation module, and denote it by $M^{\mathbf{ev}_\gamma}$ (resp. $M^{\widetilde{\mathbf{ev}}_\gamma}$).

1.6. In the rest of this section, we give a relation with the Lie algebra $\mathfrak{g}_{\hat{\mathbf{Q}}}(\mathbf{m})$ introduced in [W, Definition 2.2].

Let $\mathbf{m} = (m_1, \dots, m_r)$ be an r -tuple of positive integers such that $\sum_{k=1}^r m_k = m$. Put $\Gamma(\mathbf{m}) = \{(i, k) \mid 1 \leq i \leq m_k, 1 \leq k \leq r\}$ and $\Gamma'(\mathbf{m}) = \Gamma(\mathbf{m}) \setminus \{(m_r, r)\}$. Then

we have the bijective map

$$\zeta : \Gamma(\mathbf{m}) \rightarrow \{1, 2, \dots, m\} \text{ such that } (i, k) \mapsto \sum_{j=1}^{k-1} m_j + i.$$

For $(i, k) \in \Gamma(\mathbf{m})$ and $j \in \mathbb{Z}$ such that $1 \leq \zeta((i, k)) + j \leq m$, put $(i + j, k) = \zeta^{-1}(\zeta((i, k)) + j)$. For $(i, k) \in \Gamma'(\mathbf{m})$ and $(j, l) \in \Gamma(\mathbf{m})$, put $a'_{(j,l)(i,k)} = a'_{\zeta((j,l))\zeta((i,k))}$. Take $\widehat{\mathbf{Q}} = (\widehat{Q}_1, \dots, \widehat{Q}_{r-1}) \in \mathbb{C}^{r-1}$. Then the Lie algebra $\mathfrak{g}_{\widehat{\mathbf{Q}}}(\mathbf{m})$ in [W, Definition 2.2] is defined by the generators $\mathcal{X}_{(i,k),t}^{\pm}$, $\mathcal{I}_{(j,l),t}$ ($(i, k) \in \Gamma'(\mathbf{m})$, $(j, l) \in \Gamma(\mathbf{m})$, $t \geq 0$) together with the following defining relations:

$$\begin{aligned} [\mathcal{I}_{(i,k),s}, \mathcal{I}_{(j,l),t}] &= 0, \quad [\mathcal{I}_{(j,l),s}, \mathcal{X}_{(i,k),t}^{\pm}] = \pm a'_{(j,l)(i,k)} \mathcal{X}_{(i,k),s+t}^{\pm}, \\ [\mathcal{X}_{(i,k),t}^{+}, \mathcal{X}_{(j,l),s}^{-}] &= \delta_{(i,k)(j,l)} \begin{cases} \mathcal{J}_{(i,k),s+t} & \text{if } i \neq m_k, \\ -\widehat{Q}_k \mathcal{J}_{(m_k,k),s+t} + \mathcal{J}_{(m_k,k),s+t+1} & \text{if } i = m_k, \end{cases} \\ [\mathcal{X}_{(i,k),t}^{\pm}, \mathcal{X}_{(j,l),s}^{\pm}] &= 0 \quad \text{if } (j, l) \neq (i \pm 1, k), \\ [\mathcal{X}_{(i,k),t+1}^{+}, \mathcal{X}_{(i \pm 1, k),s}^{+}] &= [\mathcal{X}_{(i,k),t}^{+}, \mathcal{X}_{(i \pm 1, k),s+1}^{+}], \quad [\mathcal{X}_{(i,k),t+1}^{-}, \mathcal{X}_{(i \pm 1, k),s}^{-}] = [\mathcal{X}_{(i,k),t}^{-}, \mathcal{X}_{(i \pm 1, k),s+1}^{-}], \\ [\mathcal{X}_{(i,k),s}^{+}, [\mathcal{X}_{(i,k),t}^{+}, \mathcal{X}_{(i \pm 1, k),u}^{+}]] &= [\mathcal{X}_{(i,k),s}^{-}, [\mathcal{X}_{(i,k),t}^{-}, \mathcal{X}_{(i \pm 1, k),u}^{-}]] = 0, \end{aligned}$$

where we put $\mathcal{J}_{(i,k),t} = \mathcal{I}_{(i,k),t} - \mathcal{I}_{(i+1,k),t}$. Then we have the following isomorphism between $\mathfrak{gl}_m^{(\mathbf{Q})}[x]$ and $\mathfrak{g}_{\widehat{\mathbf{Q}}}(\mathbf{m})$ under the suitable choice of the deformation parameters \mathbf{Q} .

Lemma 1.7. *Assume that $\widehat{Q}_i \neq 0$ for all $i = 1, 2, \dots, r-1$. We take $\mathbf{Q} = (Q_1, Q_2, \dots, Q_{m-1}) \in \mathbb{C}^{m-1}$ as*

$$Q_i = \begin{cases} \widehat{Q}_k^{-1} & \text{if } \zeta^{-1}(i) = (m_k, k) \text{ for some } k = 1, 2, \dots, r-1, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have the isomorphism of Lie algebras $\Phi : \mathfrak{gl}_m^{(\mathbf{Q})}[x] \rightarrow \mathfrak{g}_{\widehat{\mathbf{Q}}}(\mathbf{m})$ such that

$$\begin{aligned} \mathcal{X}_{i,t}^{+} &\mapsto \begin{cases} \mathcal{X}_{\zeta^{-1}(i),t}^{+} & \text{if } \zeta^{-1}(i) \neq (m_k, k) \text{ for all } k = 1, \dots, r-1, \\ -\widehat{Q}_k^{-1} \mathcal{X}_{\zeta^{-1}(i),t}^{+} & \text{if } \zeta^{-1}(i) = (m_k, k) \text{ for some } k = 1, \dots, r-1, \end{cases} \\ \mathcal{X}_{i,t}^{-} &\mapsto \mathcal{X}_{\zeta^{-1}(i),t}^{-}, \quad \mathcal{I}_{j,t} \mapsto \mathcal{I}_{\zeta^{-1}(j),t}. \end{aligned}$$

Proof. We see the well-definedness of Φ by checking the defining relations. The inverse homomorphism of Φ is given by

$$\mathcal{X}_{(i,k),t}^{+} \mapsto \begin{cases} \mathcal{X}_{\zeta((i,k)),t}^{+} & \text{if } i \neq m_k, \\ -\widehat{Q}_k \mathcal{X}_{\zeta((i,k)),t}^{+} & \text{if } i = m_k, \end{cases} \quad \mathcal{X}_{(i,k),t}^{-} \mapsto \mathcal{X}_{\zeta((i,k)),t}^{-}, \quad \mathcal{I}_{(j,l),t} \mapsto \mathcal{I}_{\zeta((j,l)),t}.$$

□

§ 2. REPRESENTATIONS OF $\mathfrak{sl}_m^{(\mathbf{Q})}[x]$

In this section, we give some fundamental results for finite dimensional $U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])$ -modules by using the standard argument.

2.1. Put $\mathfrak{h} = \bigoplus_{i=1}^{m-1} \mathbb{C}\mathcal{J}_{i,0} \subset \mathfrak{sl}_m^{(\mathbf{Q})}[x]$, then \mathfrak{h} is a commutative Lie subalgebra of $\mathfrak{sl}_m^{(\mathbf{Q})}[x]$. (Note that, if $Q_i = 0$ for all $i = 1, \dots, m-1$, \mathfrak{h} is a Cartan subalgebra of \mathfrak{sl}_m .) Let \mathfrak{h}^* be the dual space of \mathfrak{h} . For each $i = 1, 2, \dots, m-1$, we take $\alpha_i \in \mathfrak{h}^*$ as $\alpha_i(\mathcal{J}_{j,0}) = a_{ji}$ for $j = 1, \dots, m-1$. Put $Q^+ = \bigoplus_{i=1}^{m-1} \mathbb{Z}_{\geq 0}\alpha_i \subset \mathfrak{h}^*$. We define the partial order on \mathfrak{h}^* by $\lambda \geq \mu$ if $\lambda - \mu \in Q^+$ for $\lambda, \mu \in \mathfrak{h}$.

2.2. For $U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])$ -mdoule M , we consider the decomposition $M = \bigoplus_{\lambda \in \mathfrak{h}^*} \widetilde{M}_\lambda$, where $\widetilde{M}_\lambda = \{x \in M \mid (h - \lambda(h))^N \cdot x = 0 \text{ for } h \in \mathfrak{h} \text{ and } N \gg 0\}$, namely $M = \bigoplus_{\lambda \in \mathfrak{h}^*} \widetilde{M}_\lambda$ is the decomposition to the generalized simultaneous eigenspaces for the action of \mathfrak{h} . By the relation (L2), we have

$$\mathcal{X}_{i,t}^\pm \cdot \widetilde{M}_\lambda \subset \widetilde{M}_{\lambda \pm \alpha_i} \quad (1 \leq i \leq m-1, t \geq 0).$$

Thus, if $U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])$ -module $M \neq 0$ is finite dimensional, there exists $\lambda \in \mathfrak{h}^*$ such that $\widetilde{M}_\lambda \neq 0$ and $\mathcal{X}_{i,t}^+ \cdot \widetilde{M}_\lambda = 0$ for all $i = 1, 2, \dots, m-1$ and $t \geq 0$. On the other hand, \widetilde{M}_λ ($\lambda \in \mathfrak{h}^*$) is closed under the action of \mathfrak{n}^0 by the relation (L1). Thus, we can take a simultaneous eigenvector $v \in \widetilde{M}_\lambda$ for the action of \mathfrak{n}^0 . Then we have the following lemma.

Lemma 2.3. *For a finite dimensional $U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])$ -module $M \neq 0$, there exists $v_0 \in M$ ($v_0 \neq 0$) satisfying the following conditions:*

- (i) $\mathcal{X}_{i,t}^+ \cdot v_0 = 0$ for all $i = 1, \dots, m-1$ and $t \geq 0$,
- (ii) $\mathcal{J}_{i,t} \cdot v_0 = u_{i,t}v_0$ ($u_{i,t} \in \mathbb{C}$) for each $i = 1, \dots, m-1$ and $t \geq 0$.

Moreover, if M is simple, we have $M = U(\mathfrak{sl}_m^{(\mathbf{Q})}[x]) \cdot v_0$.

2.4. Highest weight modules. For $U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])$ -module M , we say that M is a highest weight module if there exists $v_0 \in M$ satisfying the following conditions:

- (i) M is generated by v_0 as a $U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])$ -module.
- (ii) $\mathcal{X}_{i,t}^+ \cdot v_0 = 0$ for all $i = 1, \dots, m-1$ and $t \geq 0$.
- (iii) $\mathcal{J}_{i,t} \cdot v_0 = u_{i,t}v_0$ ($u_{i,t} \in \mathbb{C}$) for each $i = 1, \dots, m-1$ and $t \geq 0$.

In this case, we say that $(u_{i,t})_{1 \leq i \leq m-1, t \geq 0} \in \prod_{i=1}^{m-1} \prod_{t \geq 0} \mathbb{C}$ is the highest weight of M , and that v_0 is a highest weight vector of M .

Let M be a highest weight $U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])$ -module with a highest weight $\mathbf{u} = (u_{i,t})_{1 \leq i \leq m-1, t \geq 0} \in \prod_{i=1}^{m-1} \prod_{t \geq 0} \mathbb{C}$ and a highest weight vector $v_0 \in M$. Thanks to the triangular decomposition (Proposition 1.4 (iv)) together with the above conditions, we have $M = U(\mathfrak{n}^-) \cdot v_0$. Let $\lambda_{\mathbf{u}} \in \mathfrak{h}^*$ be as $\lambda_{\mathbf{u}}(\mathcal{J}_{i,0}) = u_{i,0}$ for $i = 1, \dots, m-1$. By $M = U(\mathfrak{n}^-) \cdot v_0$ and the relation (L2), we have the weight space decomposition

$$(2.4.1) \quad M = \bigoplus_{\substack{\mu \in \mathfrak{h}^* \\ \mu \leq \lambda_{\mathbf{u}}}} M_\mu, \text{ where } M_\mu = \{x \in M \mid h \cdot x = \mu(h) \cdot x \text{ for } h \in \mathfrak{h}\},$$

and we also have $\dim_{\mathbb{C}} M_{\lambda_{\mathbf{u}}} = 1$.

2.5. Verma modules. For $\mathbf{u} = (u_{i,t}) \in \prod_{i=1}^{m-1} \prod_{t \geq 0} \mathbb{C}$, let $\mathfrak{J}(\mathbf{u})$ be the left ideal of $U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])$ generated by $\mathcal{X}_{i,t}^+$ ($1 \leq i \leq m-1$, $t \geq 0$) and $\mathcal{J}_{i,t} - u_{i,t}$ ($1 \leq i \leq m-1$, $t \geq 0$). We define the Verma module $\mathcal{M}(\mathbf{u}) = U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])/\mathfrak{J}(\mathbf{u})$. Then $\mathcal{M}(\mathbf{u})$ is a highest weight module of highest weight \mathbf{u} , and any highest weight module of highest weight \mathbf{u} is realized as a quotient of the Verma module $\mathcal{M}(\mathbf{u})$. By the weight space decomposition (2.4.1), we see that $\mathcal{M}(\mathbf{u})$ has the unique maximal proper submodule $\text{rad } \mathcal{M}(\mathbf{u})$. Put $\mathcal{L}(\mathbf{u}) = \mathcal{M}(\mathbf{u})/\text{rad } \mathcal{M}(\mathbf{u})$, then we have the following proposition.

Proposition 2.6. *For $\mathbf{u} = (u_{i,t}) \in \prod_{i=1}^{m-1} \prod_{t \geq 0} \mathbb{C}$, a highest weight simple $U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])$ -module of highest weight \mathbf{u} is isomorphic to $\mathcal{L}(\mathbf{u})$. Moreover, any finite dimensional simple $U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])$ -module is isomorphic to $\mathcal{L}(\mathbf{u})$ for some $\mathbf{u} = (u_{i,t}) \in \prod_{i=1}^{m-1} \prod_{t \geq 0} \mathbb{C}$.*

Proof. By Lemma 2.3, a finite dimensional simple $U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])$ -module is a highest weight module. Then we have the proposition by the above arguments. \square

§ 3. REPRESENTATIONS OF $\mathfrak{gl}_m^{(\mathbf{Q})}[x]$

For finite dimensional $U(\mathfrak{gl}_m^{(\mathbf{Q})}[x])$ -modules, we can develop a similar argument as in the case of $U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])$ discussed in the previous section. In this section, we give only some notation for $U(\mathfrak{gl}_m^{(\mathbf{Q})}[x])$ -modules.

3.1. Highest weight modules. For $U(\mathfrak{gl}_m^{(\mathbf{Q})}[x])$ -module M , we say that M is a highest weight module if there exists $v_0 \in M$ satisfying the following conditions:

- (i) M is generated by v_0 as a $U(\mathfrak{gl}_m^{(\mathbf{Q})}[x])$ -module.
- (ii) $\mathcal{X}_{i,t}^+ \cdot v_0 = 0$ for all $i = 1, \dots, m-1$ and $t \geq 0$.
- (iii) $\mathcal{I}_{j,t} \cdot v_0 = \tilde{u}_{j,t} v_0$ ($\tilde{u}_{j,t} \in \mathbb{C}$) for each $j = 1, \dots, m$ and $t \geq 0$.

In this case, we say that $(\tilde{u}_{j,t})_{1 \leq j \leq m, t \geq 0} \in \prod_{j=1}^m \prod_{t \geq 0} \mathbb{C}$ is the highest weight of M , and that v_0 is a highest weight vector of M .

3.2. Verma modules. For $\tilde{\mathbf{u}} = (\tilde{u}_{j,t}) \in \prod_{j=1}^m \prod_{t \geq 0} \mathbb{C}$, let $\mathfrak{J}(\tilde{\mathbf{u}})$ be the left ideal of $U(\mathfrak{gl}_m^{(\mathbf{Q})}[x])$ generated by $\mathcal{X}_{i,t}^+$ ($1 \leq i \leq m-1$, $t \geq 0$) and $\mathcal{I}_{j,t} - \tilde{u}_{j,t}$ ($1 \leq j \leq m$, $t \geq 0$). We define the Verma module $\mathcal{M}(\tilde{\mathbf{u}}) = U(\mathfrak{gl}_m^{(\mathbf{Q})}[x])/\mathfrak{J}(\tilde{\mathbf{u}})$. Then $\mathcal{M}(\tilde{\mathbf{u}})$ is a highest weight module of highest weight $\tilde{\mathbf{u}}$, and any highest weight module of highest weight $\tilde{\mathbf{u}}$ is realized as a quotient of the Verma module $\mathcal{M}(\tilde{\mathbf{u}})$. $\mathcal{M}(\tilde{\mathbf{u}})$ has the unique maximal proper submodule $\text{rad } \mathcal{M}(\tilde{\mathbf{u}})$. Put $\mathcal{L}(\tilde{\mathbf{u}}) = \mathcal{M}(\tilde{\mathbf{u}})/\text{rad } \mathcal{M}(\tilde{\mathbf{u}})$, then we have the following proposition.

Proposition 3.3. *For $\tilde{\mathbf{u}} = (\tilde{u}_{j,t}) \in \prod_{j=1}^m \prod_{t \geq 0} \mathbb{C}$, a highest weight simple $U(\mathfrak{gl}_m^{(\mathbf{Q})}[x])$ -module of highest weight $\tilde{\mathbf{u}}$ is isomorphic to $\mathcal{L}(\tilde{\mathbf{u}})$. Moreover, any finite dimensional simple $U(\mathfrak{gl}_m^{(\mathbf{Q})}[x])$ -module is isomorphic to $\mathcal{L}(\tilde{\mathbf{u}})$ for some $\tilde{\mathbf{u}} = (u_{j,t}) \in \prod_{j=1}^m \prod_{t \geq 0} \mathbb{C}$.*

§ 4. RANK 1 CASE ; SOME RELATIONS IN $U(\mathfrak{sl}_2^{(Q)}[x])$

4.1. Take $Q \in \mathbb{C}$, then $\mathfrak{sl}_2^{(Q)}[x]$ is a Lie algebra over \mathbb{C} generated by \mathcal{X}_t^\pm and \mathcal{J}_t ($t \in \mathbb{Z}_{\geq 0}$) together with the following defining relations:

$$\begin{aligned} \text{(L1)} \quad & [\mathcal{J}_s, \mathcal{J}_t] = 0, \\ \text{(L2)} \quad & [\mathcal{J}_s, \mathcal{X}_t^\pm] = \pm 2\mathcal{X}_{s+t}^\pm, \\ \text{(L3)} \quad & [\mathcal{X}_t^+, \mathcal{X}_s^-] = \mathcal{J}_{s+t} - Q\mathcal{J}_{s+t+1}, \\ \text{(L4)} \quad & [\mathcal{X}_t^\pm, \mathcal{X}_s^\pm] = 0. \end{aligned}$$

(In the rank 1 case, we omit the first index of the generators since it is trivial.) By checking the defining relations, we see that there exists the algebra anti-automorphism $\dagger : U(\mathfrak{sl}_2^{(Q)}[x]) \rightarrow U(\mathfrak{sl}_2^{(Q)}[x])$ such that

$$(4.1.1) \quad \dagger(\mathcal{X}_t^+) = \mathcal{X}_t^-, \quad \dagger(\mathcal{X}_t^-) = \mathcal{X}_t^+, \quad \dagger(\mathcal{J}_t) = \mathcal{J}_t.$$

Clearly, \dagger^2 is the identity on $U(\mathfrak{sl}_2^{(Q)}[x])$.

4.2. For $t, b \in \mathbb{Z}_{\geq 0}$, we define an element $\mathcal{X}_t^{\pm(b)}$ (resp. $\mathcal{X}_t^{-\langle b \rangle}$) of $U(\mathfrak{sl}_2^{(Q)}[x])$ by

$$\mathcal{X}_t^{\pm(b)} = \frac{(\mathcal{X}_t^\pm)^b}{b!}.$$

For convenience, we put $\mathcal{X}_t^{\pm(b)} = 0$ for $b \in \mathbb{Z}_{<0}$.

For $t, p, h \in \mathbb{Z}_{\geq 0}$, we define an element $\mathcal{X}_t^{+((p);h)}$ (resp. $\mathcal{X}_t^{-((p);h)}$) of $U(\mathfrak{sl}_2^{(Q)}[x])$ by

$$(4.2.1) \quad \mathcal{X}_t^{\pm((0);h)} = 1, \quad \mathcal{X}_t^{\pm((p);h)} = \sum_{w=0}^p \binom{p}{w} (-Q)^w \mathcal{X}_{t+ph+w}^\pm \quad \text{for } p > 0.$$

Clearly, we have $\dagger(\mathcal{X}_t^{+((p);h)}) = \mathcal{X}_t^{-((p);h)}$. For examples, we have

$$\begin{aligned} \mathcal{X}_t^{\pm((0);h)} &= 1, \quad \mathcal{X}_t^{\pm((1);h)} = \mathcal{X}_{t+h}^\pm + (-Q)\mathcal{X}_{t+h+1}^\pm, \\ \mathcal{X}_t^{\pm((2);h)} &= \mathcal{X}_{t+2h}^\pm + 2(-Q)\mathcal{X}_{t+2h+1}^\pm + (-Q)^2\mathcal{X}_{t+2h+2}^\pm, \\ \mathcal{X}_t^{\pm((3);h)} &= \mathcal{X}_{t+3h}^\pm + 3(-Q)\mathcal{X}_{t+3h+1}^\pm + 3(-Q)^2\mathcal{X}_{t+3h+2}^\pm + (-Q)^3\mathcal{X}_{t+3h+3}^\pm. \end{aligned}$$

For $s, p \in \mathbb{Z}_{\geq 0}$, we define an element $\mathcal{J}_s^{\langle p \rangle}$ of $U(\mathfrak{sl}_2^{(Q)}[x])$ inductively on p by

$$(4.2.2) \quad \mathcal{J}_s^{\langle 0 \rangle} = 1, \quad \mathcal{J}_s^{\langle p \rangle} = \frac{1}{p} \sum_{z=1}^p (-1)^{z-1} \left(\sum_{w=0}^z \binom{z}{w} (-Q)^w \mathcal{J}_{zs+w} \right) \mathcal{J}_s^{\langle p-z \rangle} \quad \text{for } p > 0.$$

For examples, we have

$$\begin{aligned}
\mathcal{J}_s^{(0)} &= 1, \quad \mathcal{J}_s^{(1)} = \mathcal{J}_s + (-Q)\mathcal{J}_{s+1}, \\
\mathcal{J}_s^{(2)} &= \frac{1}{2} \left((\mathcal{J}_s^2 - \mathcal{J}_{2s}) + 2(-Q)(\mathcal{J}_s\mathcal{J}_{s+1} - \mathcal{J}_{2s+1}) + (-Q)^2(\mathcal{J}_{s+1}^2 - \mathcal{J}_{2s+2}) \right), \\
\mathcal{J}_s^{(3)} &= \frac{1}{3} \left((\mathcal{J}_s^3 - 2\mathcal{J}_s\mathcal{J}_{2s} + \mathcal{J}_{3s}) + 3(-Q)(\mathcal{J}_s^2\mathcal{J}_{s+1} - \mathcal{J}_s\mathcal{J}_{2s+1} - \mathcal{J}_{s+1}\mathcal{J}_{2s} + \mathcal{J}_{3s+1}) \right. \\
&\quad \left. + (-Q)^2(3\mathcal{J}_s\mathcal{J}_{s+1}^2 - 2\mathcal{J}_s\mathcal{J}_{2s+2} - 4\mathcal{J}_{s+1}\mathcal{J}_{2s+1} + 3\mathcal{J}_{3s+2}) \right. \\
&\quad \left. + (-Q)^3(\mathcal{J}_{s+1}^3 - 2\mathcal{J}_{s+1}\mathcal{J}_{2s+2} + \mathcal{J}_{3s+3}) \right).
\end{aligned}$$

Lemma 4.3. For $s, t, p \in \mathbb{Z}_{\geq 0}$, we have the following relations in $U(\mathfrak{sl}_2^{(Q)}[x])$.

$$\begin{aligned}
\text{(i)} \quad [\mathcal{J}_s^{(p)}, \mathcal{X}_t^+] &= \sum_{z=1}^p (-1)^{z+1} (z+1) \mathcal{J}_s^{(p-z)} \mathcal{X}_t^{+((z);s)}. \\
\text{(ii)} \quad [\mathcal{J}_s^{(p)}, \mathcal{X}_t^-] &= - \sum_{z=1}^p (-1)^{z+1} (z+1) \mathcal{X}_t^{-((z);s)} \mathcal{J}_s^{(p-z)}.
\end{aligned}$$

Proof. (ii) follows from (i) by applying the algebra anti-automorphism \dagger defined in (4.1.1). Then, we prove only (i) by the induction on p .

If $p = 0$, (i) is clear. If $p > 0$, by the definition (4.2.2), we have

$$\mathcal{J}_s^{(p)} \mathcal{X}_t^+ = \frac{1}{p} \sum_{z=1}^p (-1)^{z-1} \left(\sum_{w=0}^z \binom{z}{w} (-Q)^w \mathcal{J}_{zs+w} \right) \mathcal{J}_s^{(p-z)} \mathcal{X}_t^+.$$

By the assumption of the induction, we have

$$\begin{aligned}
\mathcal{J}_s^{(p)} \mathcal{X}_t^+ &= \frac{1}{p} \sum_{z=1}^p (-1)^{z-1} \left(\sum_{w=0}^z \binom{z}{w} (-Q)^w \mathcal{J}_{zs+w} \right) \\
&\quad \times \left(\mathcal{X}_t^+ \mathcal{J}_s^{(p-z)} + \sum_{k=1}^{p-z} (-1)^{k+1} (k+1) \mathcal{J}_s^{(p-z-k)} \mathcal{X}_t^{+((k);s)} \right) \\
&= \frac{1}{p} \sum_{z=1}^p (-1)^{z-1} \sum_{w=0}^z \binom{z}{w} (-Q)^w (\mathcal{X}_t^+ \mathcal{J}_{zs+w} + 2\mathcal{X}_{t+zs+w}^+ \mathcal{J}_s^{(p-z)}) \\
&\quad + \frac{1}{p} \sum_{z=1}^p \sum_{w=0}^z \sum_{k=1}^{p-z} (-1)^{z+k} \binom{z}{w} (k+1) (-Q)^w \mathcal{J}_{zs+w} \mathcal{J}_s^{(p-z-k)} \mathcal{X}_t^{+((k);s)}.
\end{aligned}$$

Applying the assumption of the induction again, we have

(4.3.1)

$$\begin{aligned}
\mathcal{J}_s^{(p)} \mathcal{X}_t^+ &= \mathcal{X}_t^+ \frac{1}{p} \sum_{z=1}^p (-1)^{z-1} \sum_{w=0}^z \binom{z}{w} (-Q)^w \mathcal{J}_{zs+w} \mathcal{J}_s^{(p-z)} \\
&\quad + 2 \frac{1}{p} \sum_{z=1}^p (-1)^{z-1} \sum_{w=0}^z \binom{z}{w} (-Q)^w \mathcal{J}_s^{(p-z)} \mathcal{X}_{t+zs+w}^+ \\
&\quad - 2 \frac{1}{p} \sum_{z=1}^p (-1)^{z-1} \sum_{w=0}^z \binom{z}{w} (-Q)^w \sum_{k=1}^{p-z} (-1)^{k+1} (k+1) \mathcal{J}_s^{(p-z-k)} \mathcal{X}_{t+zs+w}^{+((k);s)} \\
&\quad + \frac{1}{p} \sum_{z=1}^p \sum_{w=0}^z \sum_{k=1}^{p-z} (-1)^{z+k} \binom{z}{w} (k+1) (-Q)^w \mathcal{J}_{zs+w} \mathcal{J}_s^{(p-z-k)} \mathcal{X}_t^{+((k);s)}.
\end{aligned}$$

By the definition (4.2.2), we have

$$(4.3.2) \quad \mathcal{X}_t^+ \frac{1}{p} \sum_{z=1}^p (-1)^{z-1} \sum_{w=0}^z \binom{z}{w} (-Q)^w \mathcal{J}_{zs+w} \mathcal{J}_s^{(p-z)} = \mathcal{X}_t^+ \mathcal{J}_s^{(p)}.$$

By the definition (4.2.1), we have

$$(4.3.3) \quad \sum_{z=1}^p (-1)^{z-1} \sum_{w=0}^z \binom{z}{w} (-Q)^w \mathcal{J}_s^{(p-z)} \mathcal{X}_{t+zs+w}^+ = \sum_{z=1}^p (-1)^{z-1} \mathcal{J}_s^{(p-z)} \mathcal{X}_t^{+((z);s)}.$$

Put

$$(*) = \sum_{z=1}^p (-1)^{z-1} \sum_{w=0}^z \binom{z}{w} (-Q)^w \sum_{k=1}^{p-z} (-1)^{k+1} (k+1) \mathcal{J}_s^{(p-z-k)} \mathcal{X}_{t+zs+w}^{+((k);s)}.$$

By the definition (4.2.1), we also have

$$\begin{aligned}
(*) &= \sum_{z=1}^p (-1)^{z-1} \sum_{w=0}^z \binom{z}{w} (-Q)^w \sum_{k=1}^{p-z} (-1)^{k+1} (k+1) \mathcal{J}_s^{(p-z-k)} \\
&\quad \times \left(\sum_{l=0}^k \binom{k}{l} (-Q)^l \mathcal{X}_{(t+zs+w)+ks+l}^+ \right) \\
&= \sum_{z=1}^p \sum_{k=1}^{p-z} (-1)^{z+k} (k+1) \mathcal{J}_s^{(p-(z+k))} \sum_{w=0}^z \sum_{l=0}^k \binom{z}{w} \binom{k}{l} (-Q)^{w+l} \mathcal{X}_{t+(z+k)s+(w+l)}^+
\end{aligned}$$

Put $z' = z + k$ and $w' = w + l$, we have

$$(*) = \sum_{z'=2}^p \sum_{k=1}^{z'-1} (-1)^{z'} (k+1) \mathcal{J}_s^{\langle p-z' \rangle} \sum_{w'=0}^{z'} \sum_{l=\max\{0, w'-(z'-k)\}}^{\min\{k, w'\}} \binom{z'-k}{w'-l} \binom{k}{l} (-Q)^{w'} \mathcal{X}_{t+z's+w'}^+.$$

By the induction on k , we can show that

$$(4.3.4) \quad \sum_{l=\max\{0, w'-(z'-k)\}}^{\min\{k, w'\}} \binom{z'-k}{w'-l} \binom{k}{l} = \binom{z'}{w'}.$$

Then, we have

$$(*) = \sum_{z'=2}^p (-1)^{z'} \left(\sum_{k=1}^{z'-1} (k+1) \right) \mathcal{J}_s^{\langle p-z' \rangle} \sum_{w'=0}^{z'} \binom{z'}{w'} (-Q)^{w'} \mathcal{X}_{t+z's+w'}^+,$$

and by the definition of (4.2.1), we have

$$(4.3.5) \quad (*) = \sum_{z'=2}^p (-1)^{z'} \frac{(z'-1)(z'+2)}{2} \mathcal{J}_s^{\langle p-z' \rangle} \mathcal{X}_t^{+((z');s)}.$$

By the definition (4.2.2), we have

$$\begin{aligned} (4.3.6) \quad & \sum_{z=1}^p \sum_{w=0}^z \sum_{k=1}^{p-z} (-1)^{z+k} \binom{z}{w} (k+1) (-Q)^w \mathcal{J}_{zs+w} \mathcal{J}_s^{\langle p-z-k \rangle} \mathcal{X}_t^{+((k);s)} \\ &= \sum_{k=1}^{p-1} \sum_{z=1}^{p-k} \sum_{w=0}^z (-1)^{z+k} \binom{z}{w} (k+1) (-Q)^w \mathcal{J}_{zs+w} \mathcal{J}_s^{\langle p-z-k \rangle} \mathcal{X}_t^{+((k);s)} \\ &= \sum_{k=1}^{p-1} (-1)^{k+1} (k+1) (p-k) \left(\frac{1}{p-k} \sum_{z=1}^{p-k} (-1)^{z-1} \sum_{w=0}^z \binom{z}{w} (-Q)^w \mathcal{J}_{zs+w} \mathcal{J}_s^{\langle (p-k)-z \rangle} \right) \mathcal{X}_t^{+((k);s)} \\ &= \sum_{k=1}^{p-1} (-1)^{k+1} (k+1) (p-k) \mathcal{J}_s^{\langle p-k \rangle} \mathcal{X}_t^{+((k);s)}. \end{aligned}$$

Combining (4.3.1) with (4.3.2), (4.3.3), (4.3.5) and (4.3.6), we have

$$\begin{aligned} \mathcal{J}_s^{(p)} \mathcal{X}_t^+ &= \mathcal{X}_t^+ \mathcal{J}_s^{(p)} + \frac{1}{p} \sum_{z=1}^p (-1)^{z+1} (2 + (z-1)(z+2) + (z+1)(p-z)) \mathcal{J}_s^{\langle p-z \rangle} \mathcal{X}_t^{+((z);s)} \\ &= \mathcal{X}_t^+ \mathcal{J}_s^{(p)} + \sum_{z=1}^p (-1)^{z+1} (z+1) \mathcal{J}_s^{\langle p-z \rangle} \mathcal{X}_t^{+((z);s)}. \end{aligned}$$

□

Lemma 4.4. For $s, t, h \in \mathbb{Z}_{\geq 0}$ and $p \in \mathbb{Z}_{>0}$, we have

$$[\mathcal{X}_t^+, \mathcal{X}_s^{-(p);h}] = \sum_{w=0}^p \binom{p}{w} (-Q)^w \mathcal{J}_{s+t+ph+w}^{(1)}.$$

Proof. By the definitions (4.2.1), (4.2.2) and the defining relation (L3), we have

$$\begin{aligned} \mathcal{X}_t^+ \mathcal{X}_s^{-(p);h} &= \sum_{w=0}^p \binom{p}{w} (-Q)^w \mathcal{X}_t^+ \mathcal{X}_{s+ph+w}^- \\ &= \sum_{w=0}^p \binom{p}{w} (-Q)^w (\mathcal{X}_{s+ph+w}^- \mathcal{X}_t^+ + \mathcal{J}_{s+t+ph+w} + (-Q) \mathcal{J}_{s+t+ph+w+1}) \\ &= \mathcal{X}_s^{-(p);h} \mathcal{X}_t^+ + \sum_{w=0}^p \binom{p}{w} (-Q)^w \mathcal{J}_{s+t+ph+w}^{(1)}. \end{aligned}$$

□

Lemma 4.5. For $s, t, h \in \mathbb{Z}_{\geq 0}$ and $p \in \mathbb{Z}_{>0}$, we have the following relations.

- (i) $[\mathcal{J}_s^{(1)}, \mathcal{X}_t^{+(p);h}] = 2\mathcal{X}_{s+t-h}^{+(p+1);h}.$
- (ii) $[\mathcal{J}_s^{(1)}, \mathcal{X}_t^{-(p);h}] = -2\mathcal{X}_{s+t-h}^{-(p+1);h}.$

Proof. (ii) follows from (i) by applying the algebra anti-automorphism \dagger defined in (4.1.1). Then, we prove (i).

By the definition (4.2.1), we have

$$\mathcal{J}_s^{(1)} \mathcal{X}_t^{+(p);h} = \sum_{w=0}^p \binom{p}{w} (-Q)^w \mathcal{J}_s^{(1)} \mathcal{X}_{t+ph+w}^+.$$

Applying Lemma 4.3 (i), we have

$$\mathcal{J}_s^{(1)} \mathcal{X}_t^{+(p);h} = \sum_{w=0}^p \binom{p}{w} (-Q)^w (\mathcal{X}_{t+ph+w}^+ \mathcal{J}_s^{(1)} + 2\mathcal{X}_{t+ph+w}^{+(1);s})$$

Then, by the definition (4.2.1) again, we have

$$\mathcal{J}_s^{(1)} \mathcal{X}_t^{+(p);h} = \mathcal{X}_t^{+(p);h} \mathcal{J}_s^{(1)} + 2 \sum_{w=0}^p \binom{p}{w} (-Q)^w (\mathcal{X}_{s+t+ph+w}^+ + (-Q) \mathcal{X}_{s+t+ph+w+1}^+).$$

On the other hand, we have

$$\sum_{w=0}^p \binom{p}{w} (-Q)^w (\mathcal{X}_{s+t+ph+w}^+ + (-Q) \mathcal{X}_{s+t+ph+w+1}^+)$$

$$\begin{aligned}
&= \mathcal{X}_{s+t+ph}^+ + \sum_{w=1}^p \left\{ \binom{p}{w} + \binom{p}{w-1} \right\} (-Q)^w \mathcal{X}_{s+t+ph+w}^+ + (-Q)^{p+1} \mathcal{X}_{s+t+ph+p+1}^+ \\
&= \sum_{w=0}^{p+1} \binom{p+1}{w} (-Q)^w \mathcal{X}_{s+t-h+(p+1)h+w}^+ \\
&= \mathcal{X}_{s+t-h}^{+(p+1);h}.
\end{aligned}$$

Thus, we have (i). □

Lemma 4.6. For $s, t, c \in \mathbb{Z}_{\geq 0}$, we have

$$[\mathcal{X}_t^+, \mathcal{X}_s^{-(c)}] = \mathcal{X}_s^{-(c-1)} \mathcal{J}_{s+t}^{(1)} - \mathcal{X}_s^{-(c-2)} \mathcal{X}_s^{-((1);s+t)}.$$

Proof. We prove the lemma by the induction on c . If $c = 0$, it is clear. If $c = 1$, it is the defining relation (L3). If $c > 1$, by the assumption of the induction, we have

$$\begin{aligned}
\mathcal{X}_t^+ \mathcal{X}_s^{-(c)} &= \frac{1}{c} \mathcal{X}_t^+ \mathcal{X}_s^{-(c-1)} \mathcal{X}_s^- \\
&= \frac{1}{c} (\mathcal{X}_s^{-(c-1)} \mathcal{X}_t^+ + \mathcal{X}_s^{-(c-2)} \mathcal{J}_{s+t}^{(1)} - \mathcal{X}_s^{-(c-3)} \mathcal{X}_s^{-((1);s+t)}) \mathcal{X}_s^-.
\end{aligned}$$

Then, by the defining relations (L3), (L4) and Lemma 4.3 (ii), we have

$$\begin{aligned}
\mathcal{X}_t^+ \mathcal{X}_s^{-(c)} &= \frac{1}{c} \{ \mathcal{X}_s^{-(c-1)} (\mathcal{X}_s^- \mathcal{X}_t^+ + \mathcal{J}_{s+t}^{(1)}) + \mathcal{X}_s^{-(c-2)} (\mathcal{X}_s^- \mathcal{J}_{s+t}^{(1)} - 2\mathcal{X}_s^{-((1);s+t)}) \\
&\quad - \mathcal{X}_s^{-(c-3)} \mathcal{X}_s^- \mathcal{X}_s^{-((1);s+t)} \} \\
&= \frac{1}{c} \{ c\mathcal{X}_s^{-(c)} \mathcal{X}_t^+ + \mathcal{X}_s^{-(c-1)} \mathcal{J}_{s+t}^{(1)} + (c-1)\mathcal{X}_s^{-(c-1)} \mathcal{J}_{s+t}^{(1)} \\
&\quad - 2\mathcal{X}_s^{-(c-2)} \mathcal{X}_s^{-((1);s+t)} - (c-2)\mathcal{X}_s^{-(c-2)} \mathcal{X}_s^{-((1);s+t)} \} \\
&= \mathcal{X}_s^{-(c)} \mathcal{X}_t^+ + \mathcal{X}_s^{-(c-1)} \mathcal{J}_{s+t}^{(1)} - \mathcal{X}_s^{-(c-2)} \mathcal{X}_s^{-((1);s+t)}.
\end{aligned}$$

□

4.7. A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is a non-increasing sequence of non-negative integers which has only finitely many non-zero terms. The size of a partition λ is the sum of all terms of λ , and we denote it by $|\lambda|$. Namely, we have $|\lambda| = \sum_{i \geq 1} \lambda_i$. If $|\lambda| = n$, we say that λ is a partition of n , and we denote it by $\lambda \vdash n$. The length of λ is the maximal i such that $\lambda_i \neq 0$, and we denote the length of λ by $\ell(\lambda)$. For a partition $\lambda = (\lambda_1, \lambda_2, \dots)$, let $m_j(\lambda)$ ($j \in \mathbb{Z}_{>0}$) be the multiplicity of j in λ . Then, for a partition λ and $t, h \in \mathbb{Z}_{\geq 0}$, we define an element $\mathcal{X}_t^{+(\lambda;h)}$ (resp. $\mathcal{X}_t^{-(\lambda;h)}$) of $U(\mathfrak{sl}_2^{(Q)}[x])$ by

$$(4.7.1) \quad \mathcal{X}_t^{\pm(\lambda;h)} = \prod_{j \geq 1} \frac{(\mathcal{X}_t^{\pm(j;h)})^{m_j(\lambda)}}{m_j(\lambda)!},$$

where we note the defining relation (L4). Clearly, we have $\dagger(\mathcal{X}_t^{+(\lambda;h)}) = \mathcal{X}_t^{-(\lambda;h)}$. For examples, we have

$$\begin{aligned}\mathcal{X}_t^{\pm((0);h)} &= 1, \quad \mathcal{X}_t^{\pm((1);h)} = \mathcal{X}_t^{\pm((1);h)}, \\ \mathcal{X}_t^{\pm((2);h)} &= \mathcal{X}_t^{\pm((2);h)}, \quad \mathcal{X}_t^{\pm((1,1);h)} = \frac{(\mathcal{X}_t^{\pm((1);h)})^2}{2!}, \\ \mathcal{X}_t^{\pm((3);h)} &= \mathcal{X}_t^{\pm((3);h)}, \quad \mathcal{X}_t^{\pm((2,1);h)} = \mathcal{X}_t^{\pm((2);h)} \mathcal{X}_t^{\pm((1);h)}, \quad \mathcal{X}_t^{\pm((1,1,1);h)} = \frac{(\mathcal{X}_t^{\pm((1);h)})^3}{3!}, \\ \mathcal{X}_t^{\pm((3,3,2,2,2,1,1);h)} &= \frac{(\mathcal{X}_t^{\pm((3);h)})^2}{2!} \frac{(\mathcal{X}_t^{\pm((2);h)})^3}{3!} \frac{(\mathcal{X}_t^{\pm((1);h)})^2}{2!}.\end{aligned}$$

For $t, h, k, b, p \in \mathbb{Z}_{\geq 0}$, we define an element $\mathcal{X}_t^{+(b;p|k;h)}$ (resp. $\mathcal{X}_t^{-(b;p|k;h)}$) of $U(\mathfrak{sl}_2^{(Q)}[x])$ by

$$(4.7.2) \quad \mathcal{X}_t^{\pm(b;p|k;h)} = \sum_{\lambda \vdash k} \mathcal{X}_t^{\pm(\lambda;h)} \mathcal{X}_t^{\pm(b-p-\ell(\lambda))}.$$

Note the defining relation (L4), we see that $\dagger(\mathcal{X}_t^{+(b;p|k;h)}) = \mathcal{X}_t^{-(b;p|k;h)}$. For examples, we have

$$\begin{aligned}\mathcal{X}_t^{\pm(b;p|0;h)} &= \mathcal{X}_t^{\pm(b-p)}, \quad \mathcal{X}_t^{\pm(b;p|1;h)} = \mathcal{X}_t^{\pm((1);h)} \mathcal{X}_t^{\pm(b-p-1)}, \\ \mathcal{X}_t^{\pm(b;p|2;h)} &= \mathcal{X}_t^{\pm((2);h)} \mathcal{X}_t^{\pm(b-p-1)} + \mathcal{X}_t^{\pm((1,1);h)} \mathcal{X}_t^{\pm(b-p-2)}, \\ \mathcal{X}_t^{\pm(b;p|3;h)} &= \mathcal{X}_t^{\pm((3);h)} \mathcal{X}_t^{\pm(b-p-1)} + \mathcal{X}_t^{\pm((2,1);h)} \mathcal{X}_t^{\pm(b-p-2)} + \mathcal{X}_t^{\pm((1,1,1);h)} \mathcal{X}_t^{\pm(b-p-3)}.\end{aligned}$$

For the element $\mathcal{X}_t^{\pm(b;p|k;h)} \in U(\mathfrak{sl}_2^{(Q)}[x])$, we prepare the following technical formulas.

Lemma 4.8. *For $t, h, k, b, p \in \mathbb{Z}_{\geq 0}$, we have the following equations for the element $\mathcal{X}_t^{\pm(b;p|k;h)}$ of $U(\mathfrak{sl}_2^{(Q)}[x])$.*

- (i) If $b - p < 0$, we have $\mathcal{X}_t^{\pm(b;p|k;h)} = 0$.
- (ii) If $k = 0$, we have $\mathcal{X}_t^{\pm(b;p|0;h)} = \mathcal{X}_t^{\pm(b-p)}$.
If $k = 1$, we have $\mathcal{X}_t^{\pm(b;p|1;h)} = \mathcal{X}_t^{\pm((1);h)} \mathcal{X}_t^{\pm(b-p-1)}$.
- (iii) If $p = b$, we have $\mathcal{X}_t^{\pm(b;p|k;h)} = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases}$
- (iv) If $b, p > 0$, we have $\mathcal{X}_t^{\pm(b;p|k;h)} = \mathcal{X}_t^{\pm(b-1;p-1|k;h)}$.
- (v) If $b, k > 0$, we have

$$\mathcal{X}_t^{\pm(b;p|k;h)} = \frac{1}{k} \sum_{z=1}^k z \mathcal{X}_t^{\pm((z);h)} \mathcal{X}_t^{\pm(b-1;p|k-z;h)}.$$

(vi) If $b > 0$, we have

$$(b - p + k)\mathcal{X}_t^{\pm(b;p|k;h)} = \mathcal{X}_t^{\pm}\mathcal{X}_t^{\pm(b-1;p|k;h)} + \sum_{z=1}^k (z+1)\mathcal{X}_t^{\pm((z);h)}\mathcal{X}_t^{\pm(b-1;p|k-z;h)}.$$

Proof. (i), (ii), (iii) and (iv) are clear from definitions.

We prove (v). Note that $\sum_{z \geq 1} zm_z(\lambda) = k$ for a partition λ of k . Then, by the definition (4.7.2), we have

$$\mathcal{X}_t^{\pm(b;p|k;h)} = \sum_{\lambda \vdash k} \mathcal{X}_t^{\pm(\lambda;h)} \mathcal{X}_t^{\pm(b-p-\ell(\lambda))} = \frac{1}{k} \sum_{\lambda \vdash k} \left(\sum_{z \geq 1} zm_z(\lambda) \right) \mathcal{X}_t^{\pm(\lambda;h)} \mathcal{X}_t^{\pm(b-p-\ell(\lambda))}.$$

On the other hand, by the definition (4.7.1), we have

$$\mathcal{X}_t^{\pm(\lambda;h)} = \prod_{j \geq 1} \frac{(\mathcal{X}_t^{\pm((j);h)})_{m_j(\lambda)}}{m_j(\lambda)!} = \frac{1}{m_z(\lambda)} \mathcal{X}_t^{\pm((z);h)} \frac{(\mathcal{X}_t^{\pm((z);h)})_{m_z(\lambda)-1}}{(m_z(\lambda)-1)!} \prod_{\substack{j \geq 1 \\ j \neq z}} \frac{(\mathcal{X}_t^{\pm((j);h)})_{m_j(\lambda)}}{m_j(\lambda)!}$$

for each z such that $m_z(\lambda) \neq 0$. Thus, we have

$$\begin{aligned} \mathcal{X}_t^{\pm(b;p|k;h)} &= \frac{1}{k} \sum_{\lambda \vdash k} \sum_{\substack{z \geq 1 \\ m_z(\lambda) \neq 0}} z \mathcal{X}_t^{\pm((z);h)} \frac{(\mathcal{X}_t^{\pm((z);h)})_{m_z(\lambda)-1}}{(m_z(\lambda)-1)!} \prod_{\substack{j \geq 1 \\ j \neq z}} \frac{(\mathcal{X}_t^{\pm((j);h)})_{m_j(\lambda)}}{m_j(\lambda)!} \mathcal{X}_t^{\pm(b-p-\ell(\lambda))} \\ &= \frac{1}{k} \sum_{z=1}^k z \mathcal{X}_t^{\pm((z);h)} \sum_{\substack{\lambda \vdash k \\ m_z(\lambda) \neq 0}} \frac{(\mathcal{X}_t^{\pm((z);h)})_{m_z(\lambda)-1}}{(m_z(\lambda)-1)!} \prod_{\substack{j \geq 1 \\ j \neq z}} \frac{(\mathcal{X}_t^{\pm((j);h)})_{m_j(\lambda)}}{m_j(\lambda)!} \mathcal{X}_t^{\pm(b-p-\ell(\lambda))} \\ &= \frac{1}{k} \sum_{z=1}^k z \mathcal{X}_t^{\pm((z);h)} \sum_{\mu \vdash k-z} \prod_{j \geq 1} \frac{(\mathcal{X}_t^{\pm((j);h)})_{m_j(\mu)}}{m_j(\mu)!} \mathcal{X}_t^{\pm(b-p-(\ell(\mu)+1))} \\ &= \frac{1}{k} \sum_{z=1}^k z \mathcal{X}_t^{\pm((z);h)} \sum_{\mu \vdash k-z} \mathcal{X}_t^{\pm(\mu;h)} \mathcal{X}_t^{\pm((b-1)-p-\ell(\mu))} \\ &= \frac{1}{k} \sum_{z=1}^k z \mathcal{X}_t^{\pm((z);h)} \mathcal{X}_t^{\pm(b-1;p|k-z;h)}. \end{aligned}$$

We prove (vi). By the definition (4.7.2), we have

$$\begin{aligned} (b - p + k)\mathcal{X}_t^{\pm(b;p|k;h)} &= k\mathcal{X}_t^{\pm(b;p|k;h)} + \sum_{\lambda \vdash k} \ell(\lambda) \mathcal{X}_t^{\pm(\lambda;h)} \mathcal{X}_t^{\pm(b-p-\ell(\lambda))} + \sum_{\lambda \vdash k} (b - p - \ell(\lambda)) \mathcal{X}_t^{\pm(\lambda;h)} \mathcal{X}_t^{\pm(b-p-\ell(\lambda))}. \end{aligned}$$

Note that $\ell(\lambda) = \sum_{z \geq 1} m_z(\lambda)$, $(b - p - \ell(\lambda))\mathcal{X}_t^{\pm(b-p-\ell(\lambda))} = \mathcal{X}_t^{\pm} \mathcal{X}_t^{\pm(b-p-\ell(\lambda)-1)}$ and the defining relation (L4), we have

$$\begin{aligned} & (b - p + k)\mathcal{X}_t^{\pm(b;p|k;h)} \\ &= k\mathcal{X}_t^{\pm(b;p|k;h)} + \sum_{\lambda \vdash k} \left(\sum_{z \geq 1} m_z(\lambda) \right) \mathcal{X}_t^{\pm(\lambda;h)} \mathcal{X}_t^{\pm(b-p-\ell(\lambda))} + \mathcal{X}_t^{\pm} \sum_{\lambda \vdash k} \mathcal{X}_t^{\pm(\lambda;h)} \mathcal{X}_t^{\pm(b-1-p-\ell(\lambda))}. \end{aligned}$$

In a similar argument as in the proof of (v), we have

$$\begin{aligned} & (b - p + k)\mathcal{X}_t^{\pm(b;p|k;h)} \\ &= k\mathcal{X}_t^{\pm(b;p|k;h)} + \sum_{z=1}^k \mathcal{X}_t^{\pm((z);h)} \mathcal{X}_t^{\pm(b-1;p|k-z;h)} + \mathcal{X}_t^{\pm} \sum_{\lambda \vdash k} \mathcal{X}_t^{\pm(\lambda;h)} \mathcal{X}_t^{\pm((b-1)-p-\ell(\lambda))}. \end{aligned}$$

Then, by (v) and the definition (4.7.2), we have

$$\begin{aligned} & (b - p + k)\mathcal{X}_t^{\pm(b;p|k;h)} \\ &= \sum_{z=1}^k z \mathcal{X}_t^{\pm((z);h)} \mathcal{X}_t^{\pm(b-1;p|k-z;h)} + \sum_{z=1}^k \mathcal{X}_t^{\pm((z);h)} \mathcal{X}_t^{\pm(b-1;p|k-z;h)} + \mathcal{X}_t^{\pm} \mathcal{X}_t^{\pm(b-1;p|k;h)} \\ &= \mathcal{X}_t^{\pm} \mathcal{X}_t^{\pm(b-1;p|k;h)} + \sum_{z=1}^k (z+1) \mathcal{X}_t^{\pm((z);h)} \mathcal{X}_t^{\pm(b-1;p|k-z;h)}. \end{aligned}$$

□

Lemma 4.9. For $s, t, c, p, k \in \mathbb{Z}_{\geq 0}$, we have

$$\begin{aligned} & (4.9.1) \\ & [\mathcal{X}_t^+, \mathcal{X}_s^{-(c;p|k;s+t)}] \\ &= \sum_{z=0}^k \sum_{w=0}^{k-z} \binom{k-z}{w} (-Q)^w \mathcal{X}_s^{-(c;p+1|z;s+t)} \mathcal{J}_{(k-z+1)(s+t)+w}^{\langle 1 \rangle} - (k+1) \mathcal{X}_s^{-(c;p+1|k+1;s+t)}. \end{aligned}$$

Proof. If $c = 0$, the equation (4.9.1) follows from Lemma 4.8 (i) and (iii). Then, we prove (4.9.1) by the induction on k in the case where $c > 0$.

If $k = 0$, we see that (4.9.1) is just the formula in Lemma 4.6 by Lemma 4.8 (ii).

If $k > 0$, by Lemma 4.8 (v) and the defining relation (L4), we have

$$\mathcal{X}_t^+ \mathcal{X}_s^{-(c;p|k;s+t)} = \frac{1}{k} \sum_{z=1}^k z \mathcal{X}_t^+ \mathcal{X}_s^{-(c-1;p|k-z;s+t)} \mathcal{X}_s^{-((z);s+t)}.$$

Applying the assumption of the induction, we have

$$\mathcal{X}_t^+ \mathcal{X}_s^{-(c;p|k;s+t)}$$

$$\begin{aligned}
&= \frac{1}{k} \sum_{z=1}^k z \left\{ \mathcal{X}_s^{-(c-1;p|k-z;s+t)} \mathcal{X}_t^+ \right. \\
&\quad + \sum_{y=0}^{k-z} \mathcal{X}_s^{-(c-1;p+1|y;s+t)} \left(\sum_{w=0}^{k-z-y} \binom{k-z-y}{w} (-Q)^w \mathcal{J}_{(k-z-y+1)(s+t)+w}^{(1)} \right) \\
&\quad \left. - (k-z+1) \mathcal{X}_s^{-(c-1;p+1|k-z+1;s+t)} \right\} \mathcal{X}_s^{-((z);s+t)}.
\end{aligned}$$

Applying Lemma 4.4 and Lemma 4.5 (ii), we have

$$\begin{aligned}
&\mathcal{X}_t^+ \mathcal{X}_s^{-(c;p|k;s+t)} \\
&= \frac{1}{k} \sum_{z=1}^k z \mathcal{X}_s^{-(c-1;p|k-z;s+t)} \left(\mathcal{X}_s^{-((z);s+t)} \mathcal{X}_t^+ + \sum_{w=0}^z \binom{z}{w} (-Q)^w \mathcal{J}_{(z+1)(s+t)+w}^{(1)} \right) \\
&\quad + \frac{1}{k} \sum_{z=1}^k \sum_{y=0}^{k-z} z \mathcal{X}_s^{-(c-1;p+1|y;s+t)} \sum_{w=0}^{k-z-y} \binom{k-z-y}{w} (-Q)^w \\
&\quad \times \left(\mathcal{X}_s^{-((z);s+t)} \mathcal{J}_{(k-z-y+1)(s+t)+w}^{(1)} - 2 \mathcal{X}_{s+(k-z-y)(s+t)+w}^{-((z+1);s+t)} \right) \\
&\quad - \frac{1}{k} \sum_{z=1}^k z (k-z+1) \mathcal{X}_s^{-(c-1;p+1|k-z+1;s+t)} \mathcal{X}_s^{-((z);s+t)}.
\end{aligned}$$

Put

$$\begin{aligned}
(*1) &= \frac{1}{k} \sum_{z=1}^k z \mathcal{X}_s^{-(c-1;p|k-z;s+t)} \mathcal{X}_s^{-((z);s+t)} \mathcal{X}_t^+, \\
(*2) &= \frac{1}{k} \sum_{z=1}^k z \mathcal{X}_s^{-(c-1;p|k-z;s+t)} \sum_{w=0}^z \binom{z}{w} (-Q)^w \mathcal{J}_{(z+1)(s+t)+w}^{(1)} \\
(*3) &= \frac{1}{k} \sum_{z=1}^k \sum_{y=0}^{k-z} z \mathcal{X}_s^{-(c-1;p+1|y;s+t)} \sum_{w=0}^{k-z-y} \binom{k-z-y}{w} (-Q)^w \mathcal{X}_s^{-((z);s+t)} \mathcal{J}_{(k-z-y+1)(s+t)+w}^{(1)} \\
(*4) &= \frac{1}{k} \sum_{z=1}^k \sum_{y=0}^{k-z} z \mathcal{X}_s^{-(c-1;p+1|y;s+t)} \sum_{w=0}^{k-z-y} \binom{k-z-y}{w} (-Q)^w \mathcal{X}_{s+(k-z-y)(s+t)+w}^{-((z+1);s+t)}, \\
(*5) &= \frac{1}{k} \sum_{z=1}^k z (k-z+1) \mathcal{X}_s^{-(c-1;p+1|k-z+1;s+t)} \mathcal{X}_s^{-((z);s+t)},
\end{aligned}$$

then we have

$$(4.9.2) \quad \mathcal{X}_t^+ \mathcal{X}_s^{-(c;p|k;s+t)} = (*1) + (*2) + (*3) - 2(*4) - (*5).$$

By Lemma 4.8 (v) together with (L4), we have

$$(4.9.3) \quad (*1) = \mathcal{X}_s^{-(c;p|k;s+t)} \mathcal{X}_t^+.$$

Put $z' = k - z$ in (*2) and apply Lemma 4.8 (iv), we have

$$(4.9.4) \quad (*2) = \frac{1}{k} \sum_{z'=0}^{k-1} (k - z') \mathcal{X}_s^{-(c;p+1|z';s+t)} \sum_{w=0}^{k-z'} \binom{k - z'}{w} (-Q)^w \mathcal{J}_{(k-z'+1)(s+t)+w}^{(1)}.$$

Put $h = z + y$ in (*3), we have

$$(*3) = \frac{1}{k} \sum_{h=1}^k \sum_{z=1}^h z \mathcal{X}_s^{-(c-1;p+1|h-z;s+t)} \mathcal{X}_s^{-((z);s+t)} \sum_{w=0}^{k-h} \binom{k-h}{w} (-Q)^w \mathcal{J}_{(k-h+1)(s+t)+w}^{(1)}.$$

Applying Lemma 4.8 (v) together with (L4), we have

$$(4.9.5) \quad (*3) = \frac{1}{k} \sum_{h=1}^k h \mathcal{X}_s^{-(c;p+1|h;s+t)} \sum_{w=0}^{k-h} \binom{k-h}{w} (-Q)^w \mathcal{J}_{(k-h+1)(s+t)+w}^{(1)}.$$

By (4.9.4) and (4.9.5), we have

$$(4.9.6) \quad (*2) + (*3) = \sum_{z=0}^k \sum_{w=0}^{k-z} \binom{k-z}{w} (-Q)^w \mathcal{X}_s^{-(c;p+1|z;s+t)} \mathcal{J}_{(k-z+1)(s+t)+w}^{(1)}.$$

We also have

$$(*4) = \frac{1}{k} \sum_{y=0}^{k-1} \sum_{z=1}^{k-y} z \mathcal{X}_s^{-(c-1;p+1|y;s+t)} \sum_{w=0}^{k-z-y} \binom{k-z-y}{w} (-Q)^w \mathcal{X}_{s+(k-z-y)(s+t)+w}^{-((z+1);s+t)}.$$

Put $h = k - y + 1$, we have

$$(*4) = \frac{1}{k} \sum_{h=2}^{k+1} \sum_{z=1}^{h-1} z \mathcal{X}_s^{-(c-1;p+1|k-h+1;s+t)} \sum_{w=0}^{h-z-1} \binom{h-z-1}{w} (-Q)^w \mathcal{X}_{s+(h-z-1)(s+t)+w}^{-((z+1);s+t)}.$$

Put

$$(\sharp) = \sum_{w=0}^{h-z-1} \binom{h-z-1}{w} (-Q)^w \mathcal{X}_{s+(h-z-1)(s+t)+w}^{-((z+1);s+t)}.$$

By (4.2.1), we have

$$(\sharp) = \sum_{w=0}^{h-z-1} \binom{h-z-1}{w} (-Q)^w \sum_{y=0}^{z+1} \binom{z+1}{y} (-Q)^y \mathcal{X}_{s+h(s+t)+w+y}^-.$$

Put $y' = w + y$, we have

$$(\sharp) = \sum_{y'=0}^h \left(\sum_{w=\max\{0, y'-(z+1)\}}^{\min\{h-(z+1), y'\}} \binom{h-(z+1)}{w} \binom{z+1}{y'-w} \right) (-Q)^{y'} \mathcal{X}_{s+h(s+t)+y'}^-.$$

Note that $\sum_{w=\max\{0, y'-(z+1)\}}^{\min\{h-(z+1), y'\}} \binom{h-(z+1)}{w} \binom{z+1}{y'-w} = \binom{h}{y'}$ by (4.3.4), we have

$$(\sharp) = \sum_{y'=0}^h \binom{h}{y'} (-Q)^{y'} \mathcal{X}_{s+h(s+t)+y'}^- = \mathcal{X}_s^{-((h); s+t)}.$$

(Use (4.2.1) again.) Then, we have

$$\begin{aligned} (*4) &= \frac{1}{k} \sum_{h=2}^{k+1} \left(\sum_{z=1}^{h-1} z \right) \mathcal{X}_s^{-(c-1; p+1|k-h+1; s+t)} \mathcal{X}_s^{-((h); s+t)} \\ &= \frac{1}{k} \sum_{h=2}^{k+1} \frac{h(h-1)}{2} \mathcal{X}_s^{-(c-1; p+1|k-h+1; s+t)} \mathcal{X}_s^{-((h); s+t)}. \end{aligned}$$

Then we have

$$\begin{aligned} (4.9.7) \quad 2(*4) + (*5) &= \sum_{z=1}^{k+1} z \mathcal{X}_s^{-(c-1; p+1|k-z+1; s+t)} \mathcal{X}_s^{-((z); s+t)} \\ &= (k+1) \mathcal{X}_s^{-(c; p+1|k+1; s+t)}, \end{aligned}$$

where the last equation follows from Lemma 4.8 (v).

By (4.9.2), (4.9.3), (4.9.6) and (4.9.7), we have

$$\begin{aligned} &\mathcal{X}_t^+ \mathcal{X}_s^{-(c; p|k; s+t)} \\ &= \mathcal{X}_s^{-(c; p|k; s+t)} \mathcal{X}_t^+ + \sum_{z=0}^k \sum_{w=0}^{k-z} \binom{k-z}{w} (-Q)^w \mathcal{X}_s^{-(c; p+1|z; s+t)} \mathcal{J}_{(k-z+1)(s+t)+w}^{(1)} \\ &\quad - (k+1) \mathcal{X}_s^{-(c; p+1|k+1; s+t)}. \end{aligned}$$

□

Proposition 4.10. *For $s, t, b, c \in \mathbb{Z}_{\geq 0}$, we have*

$$(4.10.1) \quad [\mathcal{X}_t^{+(b)}, \mathcal{X}_s^{-(c)}] = \sum_{p=1}^{\min\{b,c\}} \sum_{k=0}^p \sum_{l=0}^{p-k} (-1)^{k+l} \mathcal{X}_s^{-(c;p|k;s+t)} \mathcal{J}_{s+t}^{\langle p-(k+l) \rangle} \mathcal{X}_t^{+(b;p|l;s+t)}.$$

Proof. We prove (4.10.1) by the induction on b . If $b = 1$, (4.10.1) follows from Lemma 4.6 together with Lemma 4.8.

If $b > 1$, we have

$$\begin{aligned} & \mathcal{X}_t^{+(b)} \mathcal{X}_s^{-(c)} \\ &= \frac{1}{b} \mathcal{X}_t^+ \mathcal{X}_t^{(b-1)} \mathcal{X}_s^{-(c)} \\ &= \frac{1}{b} \mathcal{X}_t^+ \left(\mathcal{X}_s^{-(c)} \mathcal{X}_t^{+(b-1)} + \sum_{p=1}^{\min\{b-1,c\}} \sum_{k=0}^p \sum_{l=0}^{p-k} (-1)^{k+l} \mathcal{X}_s^{-(c;p|k;s+t)} \mathcal{J}_{s+t}^{\langle p-(k+l) \rangle} \mathcal{X}_t^{+(b-1;p|l;s+t)} \right) \end{aligned}$$

by the assumption of the induction. Applying Lemma 4.6 and Lemma 4.9, we have

$$\begin{aligned} & \mathcal{X}_t^{+(b)} \mathcal{X}_s^{-(c)} \\ &= \frac{1}{b} \left\{ \left(\mathcal{X}_s^{-(c)} \mathcal{X}_t^+ + \mathcal{X}_s^{-(c-1)} \mathcal{J}_{s+t}^{\langle 1 \rangle} - \mathcal{X}_s^{-(c-2)} \mathcal{X}_s^{-((1);s+t)} \right) \mathcal{X}_t^{+(b-1)} \right. \\ & \quad + \sum_{p=1}^{\min\{b-1,c\}} \sum_{k=0}^p \sum_{l=0}^{p-k} (-1)^{k+l} \\ & \quad \times \left(\mathcal{X}_s^{-(c;p|k;s+t)} \mathcal{X}_t^+ + \sum_{z=0}^k \sum_{w=0}^{k-z} \binom{k-z}{w} (-Q)^w \mathcal{X}_s^{-(c;p+1|z;s+t)} \mathcal{J}_{(k-z+1)(s+t)+w}^{\langle 1 \rangle} \right. \\ & \quad \left. \left. - (k+1) \mathcal{X}_s^{-(c;p+1|k+1;s+t)} \right) \mathcal{J}_{s+t}^{\langle p-(k+l) \rangle} \mathcal{X}_t^{+(b-1;p|l;s+t)} \right\}. \end{aligned}$$

On the other hand, by Lemma 4.3, we have

$$\begin{aligned} & \mathcal{X}_s^{-(c;p|k;s+t)} \mathcal{X}_t^+ \mathcal{J}_{s+t}^{\langle p-(k+l) \rangle} \mathcal{X}_t^{+(b-1;p|l;s+t)} \\ &= \mathcal{X}_s^{-(c;p|k;s+t)} \left(\mathcal{J}_{s+t}^{\langle p-(k+l) \rangle} \mathcal{X}_t^+ - \sum_{z=1}^{p-(k+l)} (-1)^{z+1} (z+1) \mathcal{J}_{s+t}^{\langle p-(k+l)-z \rangle} \mathcal{X}_t^{+((z);s+t)} \right) \mathcal{X}_t^{+(b-1;p|l;s+t)}. \end{aligned}$$

Put

$$\begin{aligned} (*1) &= b \mathcal{X}_s^{-(c)} \mathcal{X}_t^{+(b)} + \mathcal{X}_s^{-(c-1)} \mathcal{J}_{s+t}^{\langle 1 \rangle} \mathcal{X}_t^{+(b-1)} - \mathcal{X}_s^{-(c-2)} \mathcal{X}_s^{-((1);s+t)} \mathcal{X}_t^{+(b-1)}, \\ (*2) &= \sum_{p=1}^{\min\{b,c\}} \sum_{k=0}^p \sum_{l=0}^{p-k} (-1)^{k+l} \mathcal{X}_s^{-(c;p|k;s+t)} \mathcal{J}_{s+t}^{\langle p-(k+l) \rangle} \mathcal{X}_t^+ \mathcal{X}_t^{+(b-1;p|l;s+t)}, \end{aligned}$$

$$\begin{aligned}
(*3) &= \sum_{p=1}^{\min\{b,c\}} \sum_{k=0}^p \sum_{l=0}^{p-k} \sum_{z=1}^{p-(k+l)} (-1)^{k+l+z} (z+1) \\
&\quad \times \mathcal{X}_s^{-(c;p|k;s+t)} \mathcal{J}_{s+t}^{\langle p-(k+l)-z \rangle} \mathcal{X}_t^{+((z);s+t)} \mathcal{X}_t^{+(b-1;p|l;s+t)}, \\
(*4) &= \sum_{p=1}^{\min\{b,c\}} \sum_{k=0}^p \sum_{l=0}^{p-k} \sum_{z=0}^k \sum_{w=0}^{k-z} (-1)^{k+l} \binom{k-z}{w} (-Q)^w \\
&\quad \times \mathcal{X}_s^{-(c;p+1|z;s+t)} \mathcal{J}_{(k-z+1)(s+t)+w}^{\langle 1 \rangle} \mathcal{J}_{s+t}^{\langle p-(k+l) \rangle} \mathcal{X}_t^{+(b-1;p|l;s+t)}, \\
(*5) &= \sum_{p=1}^{\min\{b,c\}} \sum_{k=0}^p \sum_{l=0}^{p-k} (-1)^{k+l+1} (k+1) \mathcal{X}_s^{-(c;p+1|k+1;s+t)} \mathcal{J}_{s+t}^{\langle p-(k+l) \rangle} \mathcal{X}_t^{+(b-1;p|l;s+t)},
\end{aligned}$$

then we have

$$(4.10.2) \quad \mathcal{X}_t^{+(b)} \mathcal{X}_s^{-(c)} = \frac{1}{b} \{ (*1) + (*2) + (*3) + (*4) + (*5) \},$$

where we note that $\mathcal{X}_t^{+(b-1;p|l;s+t)} = 0$ if $p = b$ by Lemma 4.8 (i).

By Lemma 4.8 (ii) together with (L4), we have

$$(4.10.3) \quad (*1) = b \mathcal{X}_s^{-(c)} \mathcal{X}_t^{+(b)} + \mathcal{X}_s^{-(c;1|0;s+t)} \mathcal{J}_{s+t}^{\langle 1 \rangle} \mathcal{X}_t^{+(b;1|0;s+t)} - \mathcal{X}_s^{-(c;1|1;s+t)} \mathcal{J}_{s+t}^{\langle 0 \rangle} \mathcal{X}_t^{+(b;1|0;s+t)}.$$

Put $l' = l + z$ in (*3), we have

$$\begin{aligned}
(*3) &= \sum_{p=1}^{\min\{b,c\}} \sum_{k=0}^p \sum_{l'=1}^{p-k} \sum_{z=1}^{l'} (-1)^{k+l'} (z+1) \mathcal{X}_s^{-(c;p|k;s+t)} \mathcal{J}_{s+t}^{\langle p-k-l' \rangle} \mathcal{X}_t^{+((z);s+t)} \mathcal{X}_t^{+(b-1;p|l'-z;s+t)}.
\end{aligned}$$

Then we have

$$\begin{aligned}
(*2) + (*3) &= \sum_{p=1}^{\min\{b,c\}} \sum_{k=0}^p \sum_{l=0}^{p-k} (-1)^{k+l} \mathcal{X}_s^{-(c;p|k;s+t)} \mathcal{J}_{s+t}^{\langle p-(k+l) \rangle} \\
&\quad \times \left(\mathcal{X}_t^{+} \mathcal{X}_t^{+(b-1;p|l;s+t)} + \sum_{z=1}^l (z+1) \mathcal{X}_t^{+((z);s+t)} \mathcal{X}_t^{+(b-1;p|l-z;s+t)} \right),
\end{aligned}$$

where we note that $\sum_{z=1}^0 \mathcal{X}_t^{+((z);s+t)} \mathcal{X}_t^{+(b-1;p|l-z;s+t)} = 0$. Applying Lemma 4.8 (vi), we have

(4.10.4)

$$\begin{aligned}
(*2) + (*3) &= \sum_{p=1}^{\min\{b,c\}} \sum_{k=0}^p \sum_{l=0}^{p-k} (-1)^{k+l} (b-p+l) \mathcal{X}_s^{-(c;p|k;s+t)} \mathcal{J}_{s+t}^{\langle p-(k+l) \rangle} \mathcal{X}_t^{+(b;p|l;s+t)} \\
&= (b-1) \mathcal{X}_s^{-(c;1|0;s+t)} \mathcal{J}_{s+t}^{\langle 1 \rangle} \mathcal{X}_t^{+(b;1|0;s+t)} - b \mathcal{X}_s^{-(c;1|0;s+t)} \mathcal{J}_{s+t}^{\langle 0 \rangle} \mathcal{X}_t^{+(b;1|1;s+t)} \\
&\quad - (b-1) \mathcal{X}_s^{-(c;1|1;s+t)} \mathcal{J}_{s+t}^{\langle 0 \rangle} \mathcal{X}_t^{+(b;1|0;s+t)} \\
&\quad + \sum_{p=2}^{\min\{b,c\}} \sum_{k=0}^p \sum_{l=0}^{p-k} (-1)^{k+l} (b-p+l) \mathcal{X}_s^{-(c;p|k;s+t)} \mathcal{J}_{s+t}^{\langle p-(k+l) \rangle} \mathcal{X}_t^{+(b;p|l;s+t)}.
\end{aligned}$$

Put $p' = p + 1$ in (*4), we have

$$\begin{aligned}
(*4) &= \sum_{p'=2}^{\min\{b,c\}} \sum_{k=0}^{p'-1} \sum_{l=0}^{p'-k-1} \sum_{z=0}^k \sum_{w=0}^{k-z} (-1)^{k+l} \binom{k-z}{w} (-Q)^w \\
&\quad \times \mathcal{X}_s^{-(c;p'|z;s+t)} \mathcal{J}_{(k-z+1)(s+t)+w}^{\langle 1 \rangle} \mathcal{J}_{s+t}^{\langle p'-(k+l)-1 \rangle} \mathcal{X}_t^{+(b-1;p'-1|l;s+t)},
\end{aligned}$$

where we note that $\mathcal{X}_t^{+(b-1;p'-1|l;s+t)} = 0$ if $p' = b+1$, and $\mathcal{X}_s^{-(c;p'|z;s+t)} = 0$ if $p' = c+1$ by Lemma 4.8 (i). Note that

$$\sum_{k=0}^{p-1} \sum_{l=0}^{p-k-1} \sum_{z=0}^k = \sum_{z=0}^{p-1} \sum_{k=z}^{p-1} \sum_{l=0}^{p-k-1} = \sum_{z=0}^{p-1} \sum_{l=0}^{p-z-1} \sum_{k=z}^{p-l-1},$$

we have

$$\begin{aligned}
(*4) &= \sum_{p=2}^{\min\{b,c\}} \sum_{z=0}^{p-1} \sum_{l=0}^{p-z-1} \mathcal{X}_s^{-(c;p|z;s+t)} \\
&\quad \times \left(\sum_{k=z}^{p-l-1} \sum_{w=0}^{k-z} (-1)^{k+l} \binom{k-z}{w} (-Q)^w \mathcal{J}_{(k-z+1)(s+t)+w}^{\langle 1 \rangle} \mathcal{J}_{s+t}^{\langle p-(k+l)-1 \rangle} \right) \mathcal{X}_t^{+(b-1;p-1|l;s+t)}.
\end{aligned}$$

Put $k' = k - z + 1$, we have

$$\begin{aligned}
&\sum_{k=z}^{p-l-1} \sum_{w=0}^{k-z} (-1)^{k+l} \binom{k-z}{w} (-Q)^w \mathcal{J}_{(k-z+1)(s+t)+w}^{\langle 1 \rangle} \mathcal{J}_{s+t}^{\langle p-(k+l)-1 \rangle} \\
&= \sum_{k'=1}^{p-l-z} \sum_{w=0}^{k'-1} (-1)^{k'+z+l-1} \binom{k'-1}{w} (-Q)^w \mathcal{J}_{k'(s+t)+w}^{\langle 1 \rangle} \mathcal{J}_{s+t}^{\langle p-k'-z-l \rangle}.
\end{aligned}$$

Since $\mathcal{J}_{k'(s+t)+w}^{(1)} = \mathcal{J}_{k'(s+t)+w} + (-Q)\mathcal{J}_{k'(s+t)+w+1}$, we see that

$$\sum_{w=0}^{k'-1} \binom{k'-1}{w} (-Q)^w \mathcal{J}_{k'(s+t)+w}^{(1)} = \sum_{w=0}^{k'} \binom{k'}{w} (-Q)^w \mathcal{J}_{k'(s+t)+w}.$$

Thus we have

$$\begin{aligned} & \sum_{k=z}^{p-l-1} \sum_{w=0}^{k-z} (-1)^{k+l} \binom{k-z}{w} (-Q)^w \mathcal{J}_{(k-z+1)(s+t)+w}^{(1)} \mathcal{J}_{s+t}^{\langle p-(k+l)-1 \rangle} \\ &= (-1)^{z+l} \sum_{k'=1}^{p-l-z} (-1)^{k'-1} \sum_{w=0}^{k'} \binom{k'}{w} (-Q)^w \mathcal{J}_{k'(s+t)+w} \mathcal{J}_{s+t}^{\langle p-k'-z-l \rangle} \\ &= (-1)^{z+l} (p-l-z) \mathcal{J}_{s+t}^{\langle p-l-z \rangle}, \end{aligned}$$

where the last equation follows from (4.2.2). Then we have

$$\begin{aligned} (*4) &= \sum_{p=2}^{\min\{b,c\}} \sum_{z=0}^{p-1} \sum_{l=0}^{p-z-1} (p-l-z) (-1)^{z+l} \mathcal{X}_s^{-(c;p|z;s+t)} \mathcal{J}_{s+t}^{\langle p-l-z \rangle} \mathcal{X}_t^{+(b-1;p-1|l;s+t)} \\ &= \sum_{p=2}^{\min\{b,c\}} \sum_{z=0}^p \sum_{l=0}^{p-z} (p-l-z) (-1)^{z+l} \mathcal{X}_s^{-(c;p|z;s+t)} \mathcal{J}_{s+t}^{\langle p-l-z \rangle} \mathcal{X}_t^{+(b-1;p-1|l;s+t)}. \end{aligned}$$

Applying Lemma 4.8 (iv), we have

$$(4.10.5) \quad (*4) = \sum_{p=2}^{\min\{b,c\}} \sum_{z=0}^p \sum_{l=0}^{p-z} (p-l-z) (-1)^{z+l} \mathcal{X}_s^{-(c;p|z;s+t)} \mathcal{J}_{s+t}^{\langle p-l-z \rangle} \mathcal{X}_t^{+(b;p|l;s+t)}.$$

Put $p' = p + 1$ in (*5), we have

$$(*5) = \sum_{p'=2}^{\min\{b,c\}} \sum_{k=0}^{p'-1} \sum_{l=0}^{p'-k-1} (-1)^{k+l+1} (k+1) \mathcal{X}_s^{-(c;p'|k+1;s+t)} \mathcal{J}_{s+t}^{\langle p'-k-l-1 \rangle} \mathcal{X}_t^{+(b-1;p'-1|l;s+t)},$$

where we note that $\mathcal{X}_t^{+(b-1;p'-1|l;s+t)} = 0$ if $p' = b + 1$, and $\mathcal{X}_s^{-(c;p'|k+1;s+t)} = 0$ if $p' = c + 1$ by Lemma 4.8 (i). Put $k' = k + 1$, we have

$$\begin{aligned} (*5) &= \sum_{p'=2}^{\min\{b,c\}} \sum_{k'=1}^{p'} \sum_{l=0}^{p'-k'} (-1)^{k'+l} k' \mathcal{X}_s^{-(c;p'|k';s+t)} \mathcal{J}_{s+t}^{\langle p'-k'-l \rangle} \mathcal{X}_t^{+(b-1;p'-1|l;s+t)} \\ &= \sum_{p=2}^{\min\{b,c\}} \sum_{k=0}^p \sum_{l=0}^{p-k} k (-1)^{k+l} \mathcal{X}_s^{-(c;p|k;s+t)} \mathcal{J}_{s+t}^{\langle p-k-l \rangle} \mathcal{X}_t^{+(b-1;p-1|l;s+t)}. \end{aligned}$$

Applying Lemma 4.8 (iv), we have

$$(4.10.6) \quad (*5) = \sum_{p=2}^{\min\{b,c\}} \sum_{k=0}^p \sum_{l=0}^{p-k} k(-1)^{k+l} \mathcal{X}_s^{-(c;p|k;s+t)} \mathcal{J}_{s+t}^{(p-k-l)} \mathcal{X}_t^{+(b;p|l;s+t)}.$$

By (4.10.2), (4.10.3), (4.10.4), (4.10.5) and (4.10.6), we have

$$\mathcal{X}_t^{+(b)} \mathcal{X}_s^{-(c)} = \mathcal{X}_s^{-(c)} \mathcal{X}_t^{+(b)} + \sum_{p=1}^{\min\{b,c\}} \sum_{k=0}^p \sum_{l=0}^{p-k} (-1)^{k+l} \mathcal{X}_s^{-(c;p|k;s+t)} \mathcal{J}_{s+t}^{(p-(k+l))} \mathcal{X}_t^{+(b;p|l;s+t)}.$$

□

§ 5. RANK 1 CASE ; FINITE DIMENSIONAL SIMPLE MODULES OF $U(\mathfrak{sl}_2^{(Q)}[x])$

In this section, we classify the finite dimensional simple $U(\mathfrak{sl}_2^{(Q)}[x])$ -modules.

5.1. 1-dimensional representations. First, we consider 1-dimensional representations of $\mathfrak{sl}_2^{(Q)}[x]$. Let $L = \mathbb{C}v$ be a 1-dimensional $U(\mathfrak{sl}_2^{(Q)}[x])$ -module with a basis $\{v\}$, then \mathcal{J}_t ($t \geq 0$) acts on v as a scalar multiplication. If $\mathcal{X}_t^+ \cdot v \neq 0$ (resp. $\mathcal{X}_t^- \cdot v \neq 0$), then $\mathcal{X}_t^+ \cdot v$ (resp. $\mathcal{X}_t^- \cdot v$) is an eigenvector for the action of \mathcal{J}_0 whose eigenvalue is different from one of v by the defining relation (L2). This is a contradiction since L is 1-dimensional. Thus, we have $\mathcal{X}_t^\pm \cdot v = 0$ for $t \geq 0$. Moreover, by the defining relation (L3), we have $(\mathcal{J}_t - Q\mathcal{J}_{t+1}) \cdot v = (\mathcal{X}_t^+ \mathcal{X}_0^- - \mathcal{X}_0^- \mathcal{X}_t^+) \cdot v = 0$. This implies that $\mathcal{J}_t \cdot v = 0$ for $t \geq 0$ if $Q = 0$, and that $\mathcal{J}_t \cdot v = Q^{-t} \mathcal{J}_0 \cdot v$ for $t > 0$ if $Q \neq 0$.

We define the set $\mathbb{B}^{(Q)}$ by

$$\mathbb{B}^{(Q)} = \begin{cases} \{0\} & \text{if } Q = 0, \\ \mathbb{C} & \text{if } Q \neq 0. \end{cases}$$

For each $\beta \in \mathbb{B}^{(Q)}$, we can define the 1-dimensional $U(\mathfrak{sl}_2^{(Q)}[x])$ -module $\mathcal{L}^\beta = \mathbb{C}v_0$ such that

$$\mathcal{X}_t^\pm \cdot v_0 = 0, \quad \mathcal{J}_t \cdot v_0 = \begin{cases} 0 & \text{if } Q = 0, \\ Q^{-t} \beta v_0 & \text{if } Q \neq 0 \end{cases} \quad (t \in \mathbb{Z}_{\geq 0})$$

by checking the defining relations of $\mathfrak{sl}_2^{(Q)}[x]$. Note that \mathcal{L}^0 is the trivial representation. Now we obtain the following lemma.

Lemma 5.2. *Any 1-dimensional $U(\mathfrak{sl}_2^{(Q)}[x])$ -module is isomorphic to \mathcal{L}^β for some $\beta \in \mathbb{B}^{(Q)}$.*

5.3. Recall from §2, a finite dimensional simple $U(\mathfrak{sl}_2^{(Q)}[x])$ -module is isomorphic to a simple highest weight module $\mathcal{L}(\mathbf{u})$ for some highest weight $\mathbf{u} = (u_t) \in \prod_{t \geq 0} \mathbb{C}$ (Proposition 2.6), where we omit the first index for the highest weight. Then, in

order to classify the finite dimensional simple $U(\mathfrak{sl}_2^{(Q)}[x])$ -module, it is enough to classify the highest weight \mathbf{u} such that $\mathcal{L}(\mathbf{u})$ is finite dimensional.

In order to obtain a necessary condition for \mathbf{u} such that $\mathcal{L}(\mathbf{u})$ is finite dimensional, we prepare the following lemma.

Lemma 5.4. *Let M be a finite dimensional $U(\mathfrak{sl}_2^{(Q)}[x])$ -module. Take an element $v \in M$ satisfying*

$$\mathcal{X}_t^+ \cdot v = 0, \quad \mathcal{J}_t \cdot v = u_t v \quad (t \in \mathbb{Z}_{\geq 0}), \quad \mathcal{X}_0^{-(n)} \cdot v \neq 0 \text{ and } \mathcal{X}_0^{-(n+1)} \cdot v = 0$$

for some $u_t \in \mathbb{C}$ ($t \in \mathbb{Z}_{\geq 0}$) and $n \in \mathbb{Z}_{\geq 0}$. (In fact, a such element exists by Lemma 2.3.) Then, for $s, t \in \mathbb{Z}_{\geq 0}$, we have

$$\sum_{w=0}^n \binom{n}{w} (-Q)^w \mathcal{J}_{t+ns+w}^{(1)} \cdot v = \sum_{k=0}^{n-1} (-1)^{n-k+1} \left(\sum_{w=0}^k \binom{k}{w} (-Q)^w \mathcal{J}_{t+ks+w}^{(1)} \right) \mathcal{J}_s^{(n-k)} \cdot v.$$

Proof. By the assumption $\mathcal{X}_0^{-(n+1)} \cdot v = 0$ and Proposition 4.10, we have

$$\begin{aligned} (5.4.1) \quad 0 &= \mathcal{X}_s^{+(n)} \mathcal{X}_0^{-(n+1)} \cdot v \\ &= \left(\mathcal{X}_0^{-(n+1)} \mathcal{X}_s^{+(n)} + \sum_{p=1}^n \sum_{k=0}^p \sum_{l=0}^{p-k} (-1)^{k+l} \mathcal{X}_0^{-(n+1;p|k;s)} \mathcal{J}_s^{(p-(k+l))} \mathcal{X}_s^{+(n;p|l;s)} \right) \cdot v. \end{aligned}$$

By the definition, we have $\mathcal{X}_s^{+(n;p|l;s)} = \sum_{\lambda \vdash l} \mathcal{X}_s^{+(\lambda;s)} \mathcal{X}_s^{+(n-p-\ell(\lambda))}$. Thus, by the definition of $\mathcal{X}_s^{+(\lambda;s)}$ and the assumption $\mathcal{X}_t^+ \cdot v = 0$ ($t \geq 0$), we have

$$\mathcal{X}_s^{+(n;p|l;s)} \cdot v = \begin{cases} v & \text{if } l = 0 \text{ and } p = n, \\ 0 & \text{otherwise.} \end{cases}$$

Then (5.4.1) implies that

$$0 = \sum_{k=0}^n (-1)^k \mathcal{X}_0^{-(n+1;n|k;s)} \mathcal{J}_s^{(n-k)} \cdot v.$$

By the definition, we have

$$\mathcal{X}_0^{-(n+1;n|k;s)} = \sum_{\lambda \vdash k} \mathcal{X}_0^{-(\lambda;s)} \mathcal{X}_0^{-(1-\ell(\lambda))} = \begin{cases} \mathcal{X}_0^- & \text{if } k = 0, \\ \mathcal{X}_0^{-((k);s)} & \text{if } k \neq 0. \end{cases}$$

Thus, we have

$$0 = \mathcal{X}_0^- \mathcal{J}_s^{\langle n \rangle} \cdot v + \sum_{k=1}^n (-1)^k \mathcal{X}_0^{-((k);s)} \mathcal{J}_s^{\langle n-k \rangle} \cdot v.$$

By multiplying \mathcal{X}_t^+ from left to this equation, we have

$$\begin{aligned} 0 &= \mathcal{X}_t^+ \mathcal{X}_0^- \mathcal{J}_s^{\langle n \rangle} \cdot v + \sum_{k=1}^n (-1)^k \mathcal{X}_t^+ \mathcal{X}_0^{-((k);s)} \mathcal{J}_s^{\langle n-k \rangle} \cdot v \\ &= \sum_{k=0}^n (-1)^k \left(\sum_{w=0}^k \binom{k}{w} (-Q)^w \mathcal{J}_{t+ks+w}^{\langle 1 \rangle} \right) \mathcal{J}_s^{\langle n-k \rangle} \cdot v, \end{aligned}$$

where we use Lemma 4.4 and the fact $\mathcal{X}_t^+ \mathcal{J}_s^{\langle n-k \rangle} \cdot v = 0$. This implies the Lemma. \square

This Lemma implies the following proposition which gives a necessary condition for \mathbf{u} such that $\mathcal{L}(\mathbf{u})$ is finite dimensional.

Proposition 5.5. *Let M be a finite dimensional $U(\mathfrak{sl}_2^{(Q)}[x])$ -module. Take an element $v \in M$ satisfying*

$$\mathcal{X}_t^+ \cdot v = 0, \quad \mathcal{J}_t \cdot v = u_t v \quad (t \in \mathbb{Z}_{\geq 0}), \quad \mathcal{X}_0^{-(n)} \cdot v \neq 0 \text{ and } \mathcal{X}_0^{-(n+1)} \cdot v = 0$$

for some $u_t \in \mathbb{C}$ ($t \in \mathbb{Z}_{\geq 0}$) and $n \in \mathbb{Z}_{\geq 0}$.

(i) *If $Q = 0$, we have $u_0 = n$, and there exist $\gamma_1, \gamma_2, \dots, \gamma_n \in \mathbb{C}$ such that*

$$u_t = p_t(\gamma_1, \gamma_2, \dots, \gamma_n) \quad (t > 0),$$

$$\text{where } p_t(\gamma_1, \dots, \gamma_n) = \gamma_1^t + \gamma_2^t + \dots + \gamma_n^t.$$

(ii) *If $Q \neq 0$, there exist $\beta, \gamma_1, \gamma_2, \dots, \gamma_n \in \mathbb{C}$ such that*

$$u_0 = n + \beta \text{ and } u_t = p_t(\gamma_1, \gamma_2, \dots, \gamma_n) + Q^{-t} \beta \quad (t > 0),$$

$$\text{where } p_t(\gamma_1, \dots, \gamma_n) = \gamma_1^t + \gamma_2^t + \dots + \gamma_n^t.$$

Proof. (i). Assume that $Q = 0$. Then, $\mathfrak{sl}_2^{(0)}[x]$ coincides with the current Lie algebra $\mathfrak{sl}_2[x]$ of \mathfrak{sl}_2 . Moreover, the Lie subalgebra of $\mathfrak{sl}_2[x]$ generated by \mathcal{X}_0^\pm and \mathcal{J}_0 is isomorphic to \mathfrak{sl}_2 . Thus, by the representation theory of \mathfrak{sl}_2 , we have $u_0 = n$.

For u_1, \dots, u_n , there exist $\gamma_1, \dots, \gamma_n \in \mathbb{C}$ such that

$$(5.5.1) \quad u_k = p_k(\gamma_1, \dots, \gamma_n) \text{ for } k = 1, \dots, n$$

by Lemma A.2.

By the definition, we have

$$(5.5.2) \quad \mathcal{J}_1^{(k)} = \frac{1}{k} \sum_{z=1}^k (-1)^{z-1} \mathcal{J}_z \mathcal{J}_1^{(k-z)}$$

since we assume $Q = 0$. By the induction on k together with (5.5.1), (5.5.2) and (A.1.1), we can show that

$$(5.5.3) \quad \mathcal{J}_1^{(k)} \cdot v = e_k(\gamma_1, \dots, \gamma_n) v \text{ for } k = 1, \dots, n,$$

where $e_k(\gamma_1, \dots, \gamma_n) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \gamma_{i_1} \gamma_{i_2} \dots \gamma_{i_k}$.

By the induction on t , we prove that

$$(5.5.4) \quad u_t = p_t(\gamma_1, \dots, \gamma_n) \quad (t > 0).$$

If $t \leq n$, (5.5.4) follows from (5.5.1). If $t > n$, by Lemma 5.4 in the case where $s = 1$, we have

$$u_t v = \mathcal{J}_{(t-n)+n} \cdot v = \sum_{k=0}^{n-1} (-1)^{n-k+1} \mathcal{J}_{(t-n)+k} \mathcal{J}_1^{(n-k)} \cdot v.$$

By the assumption of the induction together with (5.5.3) and (A.1.2), we have

$$u_t v = \sum_{k=0}^{n-1} (-1)^{n-k+1} p_{t-n+k}(\gamma_1, \dots, \gamma_n) e_{n-k}(\gamma_1, \dots, \gamma_n) v = p_t(\gamma_1, \dots, \gamma_n) v.$$

(ii). Assume that $Q \neq 0$. For u_0, u_1, \dots, u_n , there exist $\beta, \gamma_1, \dots, \gamma_n \in \mathbb{C}$ such that

$$(5.5.5) \quad u_0 = n + \beta \text{ and } u_k = p_k(\gamma_1, \dots, \gamma_n) + Q^{-k} \beta \text{ for } k = 1, \dots, n$$

by Lemma A.2.

By the induction on k , we prove that

$$(5.5.6) \quad \mathcal{J}_0^{(k)} \cdot v = e_k(\theta_1, \theta_2, \dots, \theta_n) v \text{ for } k = 1, \dots, n,$$

where $\theta_i = 1 - Q\gamma_i$ ($1 \leq i \leq n$) and $e_k(\theta_1, \dots, \theta_n) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \theta_{i_1} \theta_{i_2} \dots \theta_{i_k}$.

In the case where $k = 1$, we have $\mathcal{J}_0^{(1)} \cdot v = (\mathcal{J}_0 + (-Q)\mathcal{J}_1) \cdot v$. Then we have $\mathcal{J}_0^{(1)} \cdot v = e_1(\theta_1, \dots, \theta_n) v$ by (5.5.5).

In the case where $1 < k \leq n$, by the definition, we have

$$\mathcal{J}_0^{(k)} \cdot v = \frac{1}{k} \sum_{z=1}^k (-1)^{z-1} \left(\sum_{w=0}^z \binom{z}{w} (-Q)^w \mathcal{J}_w \right) \mathcal{J}_0^{(k-z)} \cdot v.$$

Applying the assumption of the induction to the right-hand side, we have

$$(5.5.7) \quad \mathcal{J}_0^{(k)} \cdot v = \frac{1}{k} \sum_{z=1}^k (-1)^{z-1} e_{k-z}(\theta_1, \dots, \theta_n) \left(\sum_{w=0}^z \binom{z}{w} (-Q)^w \mathcal{J}_w \right) \cdot v,$$

where we note that $\mathcal{J}_0^{(0)} = e_0(\theta_1, \dots, \theta_n) = 1$. On the other hand, by (5.5.5), we have

$$(5.5.8) \quad \sum_{w=0}^z \binom{z}{w} (-Q)^w \mathcal{J}_w \cdot v = p_z(\theta_1, \dots, \theta_n) v \quad (1 \leq z \leq k \leq n),$$

where we note that $\sum_{w=0}^z \binom{z}{w} (-1)^w = 0$. By (5.5.7) and (5.5.8) together with (A.1.1), we have (5.5.6).

By the induction on t , we prove that

$$(5.5.9) \quad u_t = p_t(\gamma_1, \dots, \gamma_n) + Q^{-t} \beta \quad (t > 0).$$

If $t \leq n$, (5.5.9) follows from (5.5.5). If $t > n$, by Lemma 5.4 in the case where $s = 0$, we have

$$\sum_{w=0}^n \binom{n}{w} (-Q)^w \mathcal{J}_{(t-n-1)+w}^{(1)} \cdot v = \sum_{k=0}^{n-1} (-1)^{n-k+1} \left(\sum_{w=0}^k \binom{k}{w} (-Q)^w \mathcal{J}_{(t-n-1)+w}^{(1)} \right) \mathcal{J}_0^{(n-k)} \cdot v.$$

By (5.5.6), we have

$$(5.5.10) \quad \begin{aligned} & \sum_{w=0}^n \binom{n}{w} (-Q)^w \mathcal{J}_{(t-n-1)+w}^{(1)} \cdot v \\ &= \sum_{k=0}^{n-1} (-1)^{n-k+1} e_{n-k}(\theta_1, \dots, \theta_n) \left(\sum_{w=0}^k \binom{k}{w} (-Q)^w \mathcal{J}_{(t-n-1)+w}^{(1)} \right) \cdot v. \end{aligned}$$

On the other hand, for $k \geq 0$, we have

$$(5.5.11) \quad \sum_{w=0}^k \binom{k}{w} (-Q)^w \mathcal{J}_{(t-n-1)+w}^{(1)} = \sum_{w=0}^{k+1} \binom{k+1}{w} (-Q)^w \mathcal{J}_{(t-n-1)+w}$$

since $\mathcal{J}_{(t-n-1)+w}^{(1)} = \mathcal{J}_{(t-n-1)+w} + (-Q)\mathcal{J}_{(t-n-1)+w+1}$. Then, by (5.5.11) and the assumption of the induction, we have

$$(5.5.12) \quad \begin{aligned} & \sum_{w=0}^n \binom{n}{w} (-Q)^w \mathcal{J}_{(t-n-1)+w}^{(1)} \cdot v \\ &= (-Q)^{n+1} \mathcal{J}_t \cdot v + \sum_{w=0}^n \binom{n+1}{w} (-Q)^w (p_{(t-n-1)+w}(\gamma_1, \dots, \gamma_n) + Q^{-((t-n-1)+w)} \beta) v \end{aligned}$$

and

$$(5.5.13) \quad \begin{aligned} & \sum_{w=0}^k \binom{k}{w} (-Q)^w \mathcal{J}_{(t-n-1)+w}^{(1)} \cdot v \\ &= \sum_{w=0}^{k+1} \binom{k+1}{w} (-Q)^w (p_{(t-n-1)+w}(\gamma_1, \dots, \gamma_n) + Q^{-((t-n-1)+w)} \beta) v \end{aligned}$$

for $k = 0, 1, \dots, n-1$. Moreover, by the direct calculations, we have

$$(5.5.14) \quad \sum_{w=0}^{k+1} \binom{k+1}{w} (-Q)^w (p_{(t-n-1)+w}(\gamma_1, \dots, \gamma_n) + Q^{-((t-n-1)+w)} \beta) = p_{k+1}^{(\gamma)}(\theta_1, \dots, \theta_n)$$

for $k \geq 0$, where $p_{k+1}^{(\gamma)}(\theta_1, \dots, \theta_n) = \gamma_1^{t-n-1} \theta_1^{k+1} + \gamma_2^{t-n-1} \theta_2^{k+1} + \dots + \gamma_n^{t-n-1} \theta_n^{k+1}$. Then, by (5.5.10), (5.5.12), (5.5.13) and (5.5.14), we have

$$\begin{aligned} & (-Q)^{n+1} \mathcal{J}_t \cdot v - (-Q)^{n+1} (p_t(\gamma_1, \dots, \gamma_n) + Q^{-t} \beta) v + p_{n+1}^{(\gamma)}(\theta_1, \dots, \theta_n) \\ &= \sum_{k=0}^{n-1} (-1)^{n-k+1} e_{n-k}(\theta_1, \dots, \theta_n) p_{k+1}^{(\gamma)}(\theta_1, \dots, \theta_n). \end{aligned}$$

Applying (A.3.2) to the right-hand side, we have

$$\mathcal{J}_t \cdot v = (p_t(\gamma_1, \dots, \gamma_n) + Q^{-t} \beta) v.$$

□

5.6. By Lemma 5.2 and Proposition 5.5, we see that the highest weight $\mathbf{u} = (u_t)_{t \geq 0}$ of a simple highest weight $U(\mathfrak{sl}_2^{(Q)}[x])$ -module $\mathcal{L}(\mathbf{u})$ has the form

$$(5.6.1) \quad u_0 = \begin{cases} n & \text{if } Q = 0 \\ n + \beta & \text{if } Q \neq 0, \end{cases} \quad u_t = \begin{cases} p_t(\gamma_1, \gamma_2, \dots, \gamma_n) & \text{if } Q = 0, \\ p_t(\gamma_1, \gamma_2, \dots, \gamma_n) + Q^{-t} \beta & \text{if } Q \neq 0 \end{cases}$$

for some $n \in \mathbb{Z}_{\geq 0}$ and $\beta, \gamma_1, \gamma_2, \dots, \gamma_n \in \mathbb{C}$ if $\mathcal{L}(\mathbf{u})$ is finite dimensional.

Let $\mathbb{C}[x]$ be the polynomial ring over \mathbb{C} with the indeterminate variable x , and let $\mathbb{C}[x]_{\text{monic}}$ be the subset of $\mathbb{C}[x]$ consisting of monic polynomials. We define the set $\mathbb{C}[x]_{\text{monic}}^{(Q)}$ by

$$\mathbb{C}[x]_{\text{monic}}^{(Q)} = \begin{cases} \mathbb{C}[x]_{\text{monic}} & \text{if } Q = 0, \\ \{\varphi \in \mathbb{C}[x]_{\text{monic}} \mid Q^{-1} \text{ is not a root of } \varphi\} & \text{if } Q \neq 0. \end{cases}$$

Recall that

$$\mathbb{B}^{(Q)} = \begin{cases} \{0\} & \text{if } Q = 0, \\ \mathbb{C} & \text{if } Q \neq 0. \end{cases}$$

We define the map

$$(5.6.2) \quad \mathbb{C}[x]_{\text{monic}}^{(Q)} \times \mathbb{B}^{(Q)} \rightarrow \prod_{t \geq 0} \mathbb{C}, \quad (\varphi, \beta) \mapsto \mathbf{u}^{(Q)}(\varphi, \beta) = (\mathbf{u}^{(Q)}(\varphi, \beta)_t)_{t \geq 0}$$

by

$$\mathbf{u}^{(Q)}(\varphi, \beta)_t = \begin{cases} \deg \varphi + \beta & \text{if } t = 0, \\ p_t(\gamma_1, \gamma_2, \dots, \gamma_n) & \text{if } t > 0 \text{ and } Q = 0, \\ p_t(\gamma_1, \gamma_2, \dots, \gamma_n) + Q^{-t}\beta & \text{if } t > 0 \text{ and } Q \neq 0, \end{cases}$$

when $\varphi = (x - \gamma_1)(x - \gamma_2) \dots (x - \gamma_n)$. We see that the map (5.6.2) is injective, and it gives a bijection between $\mathbb{C}[x]_{\text{monic}}^{(Q)} \times \mathbb{B}^{(Q)}$ and the set of highest weight $\mathbf{u} = (u_t)_{t \geq 0}$ satisfying (5.6.1), where we note that

$$p_t(\gamma_1, \dots, \gamma_n, \underbrace{Q^{-1}, \dots, Q^{-1}}_k) + Q^{-t}\beta = p_t(\gamma_1, \dots, \gamma_n) + Q^{-t}(\beta + k).$$

Then we have the following corollary of Lemma 5.2 and Proposition 5.5.

Corollary 5.7. *Any finite dimensional simple $U(\mathfrak{sl}_2^{(Q)}[x])$ -module is isomorphic to $\mathcal{L}(\mathbf{u}^{(Q)}(\varphi, \beta))$ for some $(\varphi, \beta) \in \mathbb{C}[x]_{\text{monic}}^{(Q)} \times \mathbb{B}^{(Q)}$. Moreover, $\mathcal{L}(\mathbf{u}^{(Q)}(\varphi, \beta)) \not\cong \mathcal{L}(\mathbf{u}^{(Q)}(\varphi', \beta'))$ if $(\varphi, \beta) \neq (\varphi', \beta')$.*

5.8. Recall the evaluation modules from the paragraph 1.5. Let $L(2)$ be the two-dimensional simple $U(\mathfrak{sl}_2)$ -module, and $v_0 \in L(2)$ be a highest weight vector. We consider the evaluation module $L(2)^{\text{ev}_\gamma}$ at $\gamma \in \mathbb{C}$, then we see that

$$\mathcal{X}_t^+ \cdot v_0 = 0, \quad \mathcal{J}_t \cdot v_0 = \gamma^t v_0 \quad (t \geq 0)$$

in $L(2)^{\text{ev}_\gamma}$.

For $(\varphi = (x - \gamma_1)(x - \gamma_2) \dots (x - \gamma_n), \beta) \in \mathbb{C}[x]_{\text{monic}}^{(Q)} \times \mathbb{B}^{(Q)}$, we consider the $U(\mathfrak{sl}_2^{(Q)}[x])$ -module

$$\mathcal{N}_{(\varphi, \beta)} = L(2)^{\text{ev}_{\gamma_1}} \otimes L(2)^{\text{ev}_{\gamma_2}} \otimes \dots \otimes L(2)^{\text{ev}_{\gamma_n}} \otimes \mathcal{L}^\beta,$$

where \mathcal{L}^β is the 1-dimensional $U(\mathfrak{sl}_2^{(Q)}[x])$ -module given in the paragraph 5.1. Let $v_0^{(k)} \in L(2)^{\text{ev}_{\gamma_k}}$ ($1 \leq k \leq n$) be a highest weight vector, and $\mathcal{L}^\beta = \mathbb{C}w_0$. Put $v_{(\varphi, \beta)} = v_0^{(1)} \otimes v_0^{(2)} \otimes \dots \otimes v_0^{(n)} \otimes w_0$. Then, for $t \geq 0$, we have

$$(5.8.1) \quad \mathcal{X}_t^+ \cdot v_{(\varphi, \beta)} = 0$$

and

$$(5.8.2) \quad \mathcal{J}_t \cdot v_{(\varphi, \beta)} = \begin{cases} (n + \beta)v_{(\varphi, \beta)} & \text{if } t = 0, \\ p_t(\gamma_1, \gamma_2, \dots, \gamma_n)v_{(\varphi, \beta)} & \text{if } t > 0 \text{ and } Q = 0, \\ (p_t(\gamma_1, \gamma_2, \dots, \gamma_n) + Q^{-t}\beta)v_{(\varphi, \beta)} & \text{if } t > 0 \text{ and } Q \neq 0. \end{cases}$$

Let $\mathcal{N}'_{(\varphi, \beta)}$ be the $U(\mathfrak{sl}_2^{(Q)}[x])$ -submodule of $\mathcal{N}_{(\varphi, \beta)}$ generated by $v_{(\varphi, \beta)}$. Then (5.8.1) and (5.8.2) imply that $\mathcal{N}'_{(\varphi, \beta)}$ is a highest weight module of highest weight $\mathbf{u}^{(Q)}(\varphi, \beta)$, and $\mathcal{N}'_{(\varphi, \beta)} / \text{rad } \mathcal{N}'_{(\varphi, \beta)}$ is isomorphic to the simple highest weight module $\mathcal{L}(\mathbf{u}^{(Q)}(\varphi, \beta))$. From the construction, $\mathcal{L}(\mathbf{u}^{(Q)}(\varphi, \beta)) \cong \mathcal{N}'_{(\varphi, \beta)} / \text{rad } \mathcal{N}'_{(\varphi, \beta)}$ is finite dimensional for each $(\varphi, \beta) \in \mathbb{C}[x]_{\text{monic}}^{(Q)} \times \mathbb{B}^{(Q)}$. Combining with Corollary 5.7, we have the following classification of finite dimensional simple $U(\mathfrak{sl}_2^{(Q)}[x])$ -modules.

Theorem 5.9. *For $(\varphi, \beta) \in \mathbb{C}[x]_{\text{monic}}^{(Q)} \times \mathbb{B}^{(Q)}$, the highest weight simple $U(\mathfrak{sl}_2^{(Q)}[x])$ -module $\mathcal{L}(\mathbf{u}^{(Q)}(\varphi, \beta))$ of highest weight $\mathbf{u}^{(Q)}(\varphi, \beta)$ is finite dimensional, and we have that*

$$\mathcal{L}(\mathbf{u}^{(Q)}(\varphi, \beta)) \cong \mathcal{L}(\mathbf{u}^{(Q)}(\varphi', \beta')) \Leftrightarrow (\varphi, \beta) = (\varphi', \beta')$$

for $(\varphi, \beta), (\varphi', \beta') \in \mathbb{C}[x]_{\text{monic}}^{(Q)} \times \mathbb{B}^{(Q)}$. Moreover,

$$\{\mathcal{L}(\mathbf{u}^{(Q)}(\varphi, \beta)) \mid (\varphi, \beta) \in \mathbb{C}[x]_{\text{monic}}^{(Q)} \times \mathbb{B}^{(Q)}\}$$

gives a complete set of isomorphism classes of finite dimensional simple $U(\mathfrak{sl}_2^{(Q)}[x])$ -modules.

Remark 5.10. If $Q \neq 0$, the evaluation module $L(2)^{\text{ev}_{Q^{-1}}}$ at Q^{-1} is not simple. Recall that $L(2)$ is the two dimensional simple $U(\mathfrak{sl}_2)$ -module with a highest weight vector v_0 . Put $v_1 = f \cdot v_0$. Then we see that $U(\mathfrak{sl}_2^{(Q)}[x]) \cdot v_1 = \mathbb{C}v_1$ is a proper $U(\mathfrak{sl}_2^{(Q)}[x])$ -submodule of $L(2)^{\text{ev}_{Q^{-1}}}$. Moreover, we have $L(2)^{\text{ev}_{Q^{-1}}} / \mathbb{C}v_1 \cong \mathcal{L}^1$ and $\mathbb{C}v_1 \cong \mathcal{L}^{-1}$ as $U(\mathfrak{sl}_2^{(Q)}[x])$ -modules.

§ 6. FINITE DIMENSIONAL SIMPLE $U(\mathfrak{sl}_m^{(Q)}[x])$ -MODULES

In this section, we classify the finite dimensional simple $U(\mathfrak{sl}_m^{(Q)}[x])$ -modules. By Proposition 2.6, any finite dimensional simple $U(\mathfrak{sl}_m^{(Q)}[x])$ -module is isomorphic to the simple highest weight module $\mathcal{L}(\mathbf{u})$ of highest weight $\mathbf{u} = (u_{i,t}) \in \prod_{i=1}^{m-1} \prod_{t \geq 0} \mathbb{C}$. Thus, it is enough to classify the highest weight \mathbf{u} such that $\mathcal{L}(\mathbf{u})$ is finite dimensional.

6.1. 1-dimensional representations. First, we consider the 1-dimensional representations of $\mathfrak{sl}_m^{(Q)}[x]$. For each $i = 1, 2, \dots, m-1$, by checking the defining relations, we have the homomorphism of algebras

$$(6.1.1) \quad \iota_i : U(\mathfrak{sl}_2^{(Q_i)}[x]) \rightarrow U(\mathfrak{sl}_m^{(Q)}[x]) \text{ by } \mathcal{X}_t^\pm \mapsto \mathcal{X}_{i,t}^\pm, \mathcal{J}_t \mapsto \mathcal{J}_{i,t} \quad (t \geq 0).$$

Let $L = \mathbb{C}v$ be a 1-dimensional $U(\mathfrak{sl}_m^{(Q)}[x])$ -module. For each $i = 1, 2, \dots, m-1$, when we regard L as a $U(\mathfrak{sl}_2^{(Q_i)}[x])$ -module through the homomorphism ι_i , we see that L is isomorphic to \mathcal{L}^{β_i} for some $\beta_i \in \mathbb{B}^{(Q_i)}$ by Lemma 5.2. Thus, we have

$$(6.1.2) \quad \mathcal{X}_{i,t}^\pm \cdot v = 0, \quad \mathcal{J}_{i,t} \cdot v = \begin{cases} 0 & \text{if } Q_i = 0, \\ Q_i^{-t} \beta_i v & \text{if } Q_i \neq 0 \end{cases} \quad (1 \leq i \leq m-1, t \geq 0)$$

for some $\beta = (\beta_i)_{1 \leq i \leq m-1} \in \prod_{i=1}^{m-1} \mathbb{B}^{(Q_i)}$.

On the other hand, by checking the defining relations, we can define the 1-dimensional $U(\mathfrak{sl}_m^{(Q)}[x])$ -module $\mathcal{L}^\beta = \mathbb{C}v$ by (6.1.2) for each $\beta = (\beta_i) \in \prod_{i=1}^{m-1} \mathbb{B}^{(Q_i)}$. Now we proved the following lemma.

Lemma 6.2. *Any 1-dimensional $U(\mathfrak{sl}_m^{(Q)})$ -module is isomorphic to \mathcal{L}^β for some $\beta \in \prod_{i=1}^{m-1} \mathbb{B}^{(Q_i)}$.*

6.3. For $\mathbf{u} = (u_{i,t}) \in \prod_{i=1}^{m-1} \prod_{t \geq 0} \mathbb{C}$, let v_0 be a highest weight vector of the simple highest weight $U(\mathfrak{sl}_m^{(Q)}[x])$ -module $\mathcal{L}(\mathbf{u})$. When we regard $\mathcal{L}(\mathbf{u})$ as a $U(\mathfrak{sl}_2^{(Q_i)}[x])$ -module through the homomorphism ι_i in (6.1.1) for each $i = 1, \dots, m-1$, we see that the $U(\mathfrak{sl}_2^{(Q_i)}[x])$ -submodule of $\mathcal{L}(\mathbf{u})$ generated by v_0 is a highest weight $U(\mathfrak{sl}_2^{(Q_i)}[x])$ -module of highest weight $\mathbf{u}_i = (u_{i,t})_{t \geq 0} \in \prod_{t \geq 0} \mathbb{C}$ with the highest weight vector v_0 . Then, if $\mathcal{L}(\mathbf{u})$ is finite dimensional, we see that $\mathbf{u}_i = \mathbf{u}^{(Q_i)}(\varphi_i, \beta_i)$ for some $(\varphi_i, \beta_i) \in \mathbb{C}[x]_{\text{monic}}^{(Q_i)} \times \mathbb{B}^{(Q_i)}$ by Theorem 5.9 (or Corollary 5.7).

For $(\varphi, \beta) = ((\varphi_i, \beta_i))_{1 \leq i \leq m-1} \in \prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^{(Q_i)} \times \mathbb{B}^{(Q_i)})$, we define

$$\mathbf{u}^{(Q)}(\varphi, \beta) = (\mathbf{u}^{(Q)}(\varphi, \beta)_{i,t})_{1 \leq i \leq m-1, t \geq 0} \in \prod_{i=1}^{m-1} \prod_{t \geq 0} \mathbb{C}$$

by

$$(6.3.1) \quad \mathbf{u}^{(\mathbf{Q})}(\varphi, \beta)_{i,t} = \begin{cases} \deg \varphi_i + \beta_i & \text{if } t = 0, \\ p_t(\gamma_{i,1}, \gamma_{i,2}, \dots, \gamma_{i,n_i}) & \text{if } t > 0 \text{ and } Q_i = 0, \\ p_t(\gamma_{i,1}, \gamma_{i,2}, \dots, \gamma_{i,n_i}) + Q_i^{-t} \beta_i & \text{if } t > 0 \text{ and } Q_i \neq 0, \end{cases}$$

when $\varphi_i = (x - \gamma_{i,1})(x - \gamma_{i,2}) \dots (x - \gamma_{i,n_i})$ ($1 \leq i \leq m-1$). Then we have that

$$(\mathbf{u}^{(\mathbf{Q})}(\varphi, \beta)_{i,t})_{t \geq 0} = \mathbf{u}^{(Q_i)}(\varphi_i, \beta_i)$$

for each $i = 1, 2, \dots, m-1$. From the definition, we see that

$$\mathbf{u}^{(\mathbf{Q})}(\varphi, \beta) = \mathbf{u}^{(\mathbf{Q})}(\varphi', \beta') \Leftrightarrow (\varphi, \beta) = (\varphi', \beta')$$

for $(\varphi, \beta), (\varphi', \beta') \in \prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^{(Q_i)} \times \mathbb{B}^{(Q_i)})$. By the above argument, any finite dimensional simple $U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])$ -module is isomorphic to $\mathcal{L}(\mathbf{u}^{(\mathbf{Q})}(\varphi, \beta))$ for some $(\varphi, \beta) \in \prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^{(Q_i)} \times \mathbb{B}^{(Q_i)})$.

On the other hand, for each $(\varphi, \beta) \in \prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^{(Q_i)} \times \mathbb{B}^{(Q_i)})$, we can construct a finite dimensional highest weight $U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])$ -module of highest weight $\mathbf{u}^{(\mathbf{Q})}(\varphi, \beta)$ as follows.

Let ω_j ($1 \leq j \leq m-1$) be the fundamental weight of \mathfrak{sl}_m , and $L(\omega_j)$ be the simple highest weight $U(\mathfrak{sl}_m)$ -module of highest weight ω_j . Let $v_0 \in L(\omega_j)$ be a highest weight vector, then we have $e_i \cdot v_0 = 0$ and $H_i \cdot v_0 = \delta_{ij} v_0$ ($1 \leq i \leq m-1$) by the definition. Recall that $L(\omega_j)^{\text{ev}_\gamma}$ is the evaluation module of $L(\omega_j)$ at $\gamma \in \mathbb{C}$. From the definition, we see that

$$(6.3.2) \quad \mathcal{X}_{i,t}^+ \cdot v_0 = 0, \quad \mathcal{J}_{i,t} \cdot v_0 = \delta_{ij} \gamma^t v_0 \quad (1 \leq i \leq m-1, t \geq 0)$$

in $L(\omega_j)^{\text{ev}_\gamma}$.

For $(\varphi, \beta) = ((\varphi_i, \beta_i))_{1 \leq i \leq m-1} \in \prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^{(Q_i)} \times \mathbb{B}^{(Q_i)})$, we consider the $U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])$ -module

$$\mathcal{N}_{(\varphi, \beta)} = \left(\bigotimes_{j=1}^{m-1} \bigotimes_{k=1}^{n_j} L(\omega_j)^{\text{ev}_{\gamma_{j,k}}} \right) \otimes \mathcal{L}^\beta,$$

where n_j and $\gamma_{j,k}$ ($1 \leq k \leq n_j$) are determined by $\varphi_j = (x - \gamma_{j,1})(x - \gamma_{j,2}) \dots (x - \gamma_{j,n_j})$ for each $j = 1, 2, \dots, m-1$, and $\beta = (\beta_i)_{1 \leq i \leq m-1}$. Let $v_0^{(j,k)} \in L(\omega_j)^{\text{ev}_{\gamma_{j,k}}}$ ($1 \leq j \leq m-1, 1 \leq k \leq n_j$) be a highest weight vector, and $\mathcal{L}^\beta = \mathbb{C}w_0$. Put $v_{(\varphi, \beta)} = (\bigotimes_{j=1}^{m-1} \bigotimes_{k=1}^{n_j} v_0^{(j,k)}) \otimes w_0 \in \mathcal{N}_{(\varphi, \beta)}$, then we have

$$(6.3.3) \quad \mathcal{X}_{i,t}^+ \cdot v_{(\varphi, \beta)} = 0, \quad \mathcal{J}_{i,t} \cdot v_{(\varphi, \beta)} = \mathbf{u}^{(\mathbf{Q})}(\varphi, \beta)_{i,t} v_{(\varphi, \beta)} \quad (1 \leq i \leq m-1, t \geq 0)$$

by (6.3.2). Let $\mathcal{N}'_{(\varphi, \beta)}$ be the $U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])$ -submodule of $\mathcal{N}_{(\varphi, \beta)}$ generated by $v_{(\varphi, \beta)}$. Then (6.3.3) implies that $\mathcal{N}'_{(\varphi, \beta)}$ is a finite dimensional highest weight module of highest weight $\mathbf{u}^{(\mathbf{Q})}(\varphi, \beta)$. Then we obtain the following classification of finite dimensional simple $U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])$ -modules.

Theorem 6.4. *For $(\varphi, \beta) \in \prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^{(Q_i)} \times \mathbb{B}^{(Q_i)})$, the highest weight simple $U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])$ -module $\mathcal{L}(\mathbf{u}^{(\mathbf{Q})}(\varphi, \beta))$ of highest weight $\mathbf{u}^{(\mathbf{Q})}(\varphi, \beta)$ is finite dimensional, and we have that*

$$\mathcal{L}(\mathbf{u}^{(\mathbf{Q})}(\varphi, \beta)) \cong \mathcal{L}(\mathbf{u}^{(\mathbf{Q})}(\varphi', \beta')) \Leftrightarrow (\varphi, \beta) = (\varphi', \beta')$$

for $(\varphi, \beta), (\varphi', \beta') \in \prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^{(Q_i)} \times \mathbb{B}^{(Q_i)})$ Moreover,

$$\{\mathcal{L}(\mathbf{u}^{(\mathbf{Q})}(\varphi, \beta)) \mid (\varphi, \beta) \in \prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^{(Q_i)} \times \mathbb{B}^{(Q_i)})\}$$

gives a complete set of isomorphism classes of finite dimensional simple $U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])$ -modules.

§ 7. FINITE DIMENSIONAL SIMPLE $U(\mathfrak{gl}_m^{(\mathbf{Q})}[x])$ -MODULES

In this section, we classify the finite dimensional simple $U(\mathfrak{gl}_m^{(\mathbf{Q})}[x])$ -modules. By Proposition 1.4 (iii), $\mathfrak{sl}_m^{(\mathbf{Q})}[x]$ is a Lie subalgebra of $\mathfrak{gl}_m^{(\mathbf{Q})}[x]$. The difference of representations of $\mathfrak{gl}_m^{(\mathbf{Q})}[x]$ from one of $\mathfrak{sl}_m^{(\mathbf{Q})}[x]$ is given by the family of 1-dimensional $U(\mathfrak{gl}_m^{(\mathbf{Q})}[x])$ -modules $\{\tilde{\mathcal{L}}^{\mathbf{h}} \mid \mathbf{h} \in \prod_{t \geq 0} \mathbb{C}\}$. We remark that $\tilde{\mathcal{L}}^{\mathbf{h}}$ ($\mathbf{h} \in \prod_{t \geq 0} \mathbb{C}$) is isomorphic to the trivial representation as a $U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])$ -module when we restrict the action.

7.1. 1-dimensional representations. For $\beta = (\beta_i)_{1 \leq i \leq m-1} \in \prod_{i=1}^{m-1} \mathbb{B}^{(Q_i)}$, by checking the defining relations, we can define the 1-dimensional $U(\mathfrak{gl}_m^{(\mathbf{Q})}[x])$ -module $\tilde{\mathcal{L}}^\beta = \mathbb{C}v$ by

$$\begin{aligned} \mathcal{X}_{i,t}^\pm \cdot v &= 0, \quad \mathcal{J}_{i,t} \cdot v = \begin{cases} 0 & \text{if } Q_i = 0, \\ Q_i^{-t} \beta_i v & \text{if } Q_i \neq 0 \end{cases} \quad (1 \leq i \leq m-1, t \geq 0), \\ \mathcal{I}_{j,t} \cdot v &= \left(\sum_{k=j}^{m-1} \mathcal{J}_{k,t} \right) \cdot v \quad (1 \leq j \leq m-1, t \geq 0), \quad \mathcal{I}_{m,t} \cdot v = 0 \quad (t \geq 0). \end{aligned}$$

Note that $\mathcal{J}_{j,t} = \mathcal{I}_{j,t} - \mathcal{I}_{j+1,t}$ in $U(\mathfrak{gl}_m^{(\mathbf{Q})}[x])$, we see that $\tilde{\mathcal{L}}^\beta \cong \mathcal{L}^\beta$ as $U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])$ -modules when we restrict the action on $\tilde{\mathcal{L}}^\beta$ to $U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])$ through the injective homomorphism Υ in the proposition 1.4 (iii).

For $\mathbf{h} = (h_t)_{t \geq 0} \in \prod_{t \geq 0} \mathbb{C}$, we can also define the 1-dimensional $U(\mathfrak{gl}_m^{(\mathbf{Q})}[x])$ -module $\tilde{\mathcal{L}}^{\mathbf{h}} = \mathbb{C}v$ by

$$\mathcal{X}_{i,t}^{\pm} \cdot v = 0, \quad \mathcal{I}_{j,t} \cdot v = h_t v \quad (1 \leq i \leq m-1, 1 \leq j \leq m, t \geq 0).$$

We see that $\tilde{\mathcal{L}}^{\mathbf{h}} \cong \mathcal{L}^{\mathbf{0}}$ as $U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])$ -modules when we restrict the action on $\tilde{\mathcal{L}}^{\mathbf{h}}$ to $U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])$ where $\mathbf{0} = (0)_{1 \leq i \leq m-1} \in \prod_{i=1}^{m-1} \mathbb{B}^{(Q_i)}$ (i.e. $\mathcal{L}^{\mathbf{0}}$ is the trivial representation). Then we have the following classification of 1-dimensional $U(\mathfrak{gl}_m^{(\mathbf{Q})})$ -modules.

Lemma 7.2. *Any 1-dimensional $U(\mathfrak{gl}_m^{(\mathbf{Q})}[x])$ -module is isomorphic to $\tilde{\mathcal{L}}^{\beta} \otimes \tilde{\mathcal{L}}^{\mathbf{h}}$ for some $\beta \in \prod_{i=1}^{m-1} \mathbb{B}^{(Q_i)}$ and $\mathbf{h} \in \prod_{t \geq 0} \mathbb{C}$. We have that*

$$\tilde{\mathcal{L}}^{\beta} \otimes \tilde{\mathcal{L}}^{\mathbf{h}} \cong \tilde{\mathcal{L}}^{\beta'} \otimes \tilde{\mathcal{L}}^{\mathbf{h}'} \Leftrightarrow (\beta, \mathbf{h}) = (\beta', \mathbf{h}').$$

Moreover, we see that $\tilde{\mathcal{L}}^{\beta} \otimes \tilde{\mathcal{L}}^{\mathbf{h}} \cong \mathcal{L}^{\beta}$ as $U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])$ -modules when we restrict the action on $\tilde{\mathcal{L}}^{\beta} \otimes \tilde{\mathcal{L}}^{\mathbf{h}}$ to $U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])$.

Proof. Let $\mathcal{L} = \mathbb{C}v$ be a 1-dimensional $U(\mathfrak{gl}_m^{(\mathbf{Q})})$ -module. By restricting the action on \mathcal{L} to $U(\mathfrak{sl}_m^{(\mathbf{Q})})$ through the injective homomorphism Υ in the proposition 1.4 (iii), we have

$$(7.2.1) \quad \begin{aligned} \mathcal{X}_{i,t}^{\pm} \cdot v &= 0, \\ \mathcal{I}_{i,t} \cdot v &= (\mathcal{I}_{i,t} - \mathcal{I}_{i+1,t}) \cdot v = \begin{cases} 0 & \text{if } Q_i = 0, \\ Q_i^{-t} \beta_i v & \text{if } Q_i \neq 0 \end{cases} \quad (1 \leq i \leq m-1, t \geq 0) \end{aligned}$$

for some $\beta = (\beta_i) \in \prod_{i=1}^{m-1} \mathbb{B}^{(Q_i)}$ by Lemma 6.2.

On the other hand, for $t \in \mathbb{Z}_{\geq 0}$, there exists $h_t \in \mathbb{C}$ such that

$$(7.2.2) \quad \mathcal{I}_{m,t} \cdot v = h_t v$$

since $\dim \mathcal{L} = 1$. Then (7.2.1) and (7.2.2) imply that

$$\mathcal{I}_{j,t} \cdot v = \left(\sum_{k=j}^{m-1} \mathcal{I}_{k,t} + h_t \right) \cdot v \quad (1 \leq j \leq m-1, t \geq 0), \quad \mathcal{I}_{m,t} \cdot v = h_t v \quad (t \geq 0).$$

Then we see that $\mathcal{L} \cong \tilde{\mathcal{L}}^{\beta} \otimes \tilde{\mathcal{L}}^{\mathbf{h}}$. The remaining statements are clear. \square

7.3. For $\tilde{\mathbf{u}} = (\tilde{u}_{j,t}) \in \prod_{j=1}^m \prod_{t \geq 0} \mathbb{C}$, let v_0 be a highest weight vector of the simple highest weight $U(\mathfrak{gl}_m^{(\mathbf{Q})}[x])$ -module $\mathcal{L}(\tilde{\mathbf{u}})$. By restricting the action on $\mathcal{L}(\tilde{\mathbf{u}})$ to $U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])$, Theorem 6.4 implies that

$$(7.3.1) \quad \tilde{u}_{i,t} - \tilde{u}_{i+1,t} = \mathbf{u}^{(\mathbf{Q})}(\varphi, \beta)_{i,t} \quad (1 \leq i \leq m-1, t \geq 0)$$

for some $(\varphi, \beta) \in \prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^{\langle Q_i \rangle} \times \mathbb{B}^{\langle Q_i \rangle})$ if $\mathcal{L}(\tilde{\mathbf{u}})$ is finite dimensional.

For $t \in \mathbb{Z}_{\geq 0}$, let $h_t \in \mathbb{C}$ be such that

$$(7.3.2) \quad \tilde{u}_{m,t} = h_t.$$

By (7.3.1) and (7.3.2), we have

$$\tilde{u}_{j,t} = \sum_{k=j}^{m-1} \mathbf{u}^{\langle \mathbf{Q} \rangle}(\varphi, \beta)_{k,t} + h_t \quad (1 \leq j \leq m-1, t \geq 0), \quad \tilde{u}_{m,t} = h_t \quad (t \geq 0)$$

for some $(\varphi, \beta) \in \prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^{\langle Q_i \rangle} \times \mathbb{B}^{\langle Q_i \rangle})$ and $\mathbf{h} = (h_t) \in \prod_{t \geq 0} \mathbb{C}$ if $\mathcal{L}(\tilde{\mathbf{u}})$ is finite dimensional.

For $(\varphi, \beta, \mathbf{h}) = ((\varphi_i, \beta_i)_{1 \leq i \leq m-1}, (h_t)_{t \geq 0}) \in \prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^{\langle Q_i \rangle} \times \mathbb{B}^{\langle Q_i \rangle}) \times \prod_{t \geq 0} \mathbb{C}$, we define

$$\tilde{\mathbf{u}}^{\langle \mathbf{Q} \rangle}(\varphi, \beta, \mathbf{h}) = (\tilde{\mathbf{u}}^{\langle \mathbf{Q} \rangle}(\varphi, \beta, \mathbf{h})_{j,t}) \in \prod_{j=1}^m \prod_{t \geq 0} \mathbb{C}$$

by

$$\tilde{\mathbf{u}}^{\langle \mathbf{Q} \rangle}(\varphi, \beta, \mathbf{h})_{j,t} = \begin{cases} \sum_{k=j}^{m-1} \mathbf{u}^{\langle \mathbf{Q} \rangle}(\varphi, \beta)_{k,t} + h_t & \text{if } 1 \leq j \leq m-1 \text{ and } t \geq 0, \\ h_t & \text{if } j = m \text{ and } t \geq 0. \end{cases}$$

From the definition, we see that

$$\tilde{\mathbf{u}}^{\langle \mathbf{Q} \rangle}(\varphi, \beta, \mathbf{h}) = \tilde{\mathbf{u}}^{\langle \mathbf{Q} \rangle}(\varphi', \beta', \mathbf{h}') \Leftrightarrow (\varphi, \beta, \mathbf{h}) = (\varphi', \beta', \mathbf{h}')$$

for $(\varphi, \beta, \mathbf{h}), (\varphi', \beta', \mathbf{h}') \in \prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^{\langle Q_i \rangle} \times \mathbb{B}^{\langle Q_i \rangle}) \times \prod_{t \geq 0} \mathbb{C}$. By the above argument, any finite dimensional simple $U(\mathfrak{gl}_m^{\langle \mathbf{Q} \rangle}[x])$ -module is isomorphic to $\mathcal{L}(\tilde{\mathbf{u}}^{\langle \mathbf{Q} \rangle}(\varphi, \beta, \mathbf{h}))$ for some $(\varphi, \beta, \mathbf{h}) \in \prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^{\langle Q_i \rangle} \times \mathbb{B}^{\langle Q_i \rangle}) \times \prod_{t \geq 0} \mathbb{C}$.

On the other hand, for each $(\varphi, \beta, \mathbf{h}) \in \prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^{\langle Q_i \rangle} \times \mathbb{B}^{\langle Q_i \rangle}) \times \prod_{t \geq 0} \mathbb{C}$, we can construct a finite dimensional highest weight $U(\mathfrak{gl}_m^{\langle \mathbf{Q} \rangle}[x])$ -module of highest weight $\tilde{\mathbf{u}}^{\langle \mathbf{Q} \rangle}(\varphi, \beta, \mathbf{h})$ as follows.

Let $P = \bigoplus_{i=1}^m \mathbb{Z}\varepsilon_i$ be the weight lattice of \mathfrak{gl}_m . Put $\tilde{\omega}_l = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_l$ for $l = 1, 2, \dots, m-1$. Let $L(\tilde{\omega}_l)$ be the simple highest weight $U(\mathfrak{gl}_m)$ -module of highest weight $\tilde{\omega}_l$, and $v_0 \in L(\tilde{\omega}_l)$ be a highest weight vector. Then, we have

$$e_i \cdot v_0 = 0 \quad (1 \leq i \leq m-1) \text{ and } K_j \cdot v_0 = \begin{cases} v_0 & \text{if } 1 \leq j \leq l, \\ 0 & \text{if } l < j \leq m. \end{cases}$$

Recall that $L(\tilde{\omega}_l)^{\tilde{\mathbf{ev}}_\gamma}$ is the evaluation module of $L(\tilde{\omega}_l)$ at $\gamma \in \mathbb{C}$. From the definition, we see that

(7.3.3)

$$\mathcal{X}_{i,t}^+ \cdot v_0 = 0 \quad (1 \leq i \leq m-1, t \geq 0), \quad \mathcal{I}_{j,t} \cdot v_0 = \begin{cases} \gamma^t v_0 & \text{if } 1 \leq j \leq l, \\ 0 & \text{if } l < j \leq m \end{cases} \quad (t \geq 0)$$

in $L(\tilde{\omega}_l)^{\tilde{\mathbf{ev}}_\gamma}$. (We remark that $L(\tilde{\omega}_l)^{\tilde{\mathbf{ev}}_\gamma} \cong L(\omega_l)^{\mathbf{ev}_\gamma}$ as $U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])$ -modules when we restrict the action on $L(\tilde{\omega}_l)^{\tilde{\mathbf{ev}}_\gamma}$ to $U(\mathfrak{sl}_m^{(\mathbf{Q})}[x])$.)

For $(\varphi, \beta, \mathbf{h}) = ((\varphi_i, \beta_i)_{1 \leq i \leq m-1}, (h_t)_{t \geq 0}) \in \prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^{(Q_i)} \times \mathbb{B}^{(Q_i)}) \times \prod_{t \geq 0} \mathbb{C}$, we consider the $U(\mathfrak{gl}_m^{(\mathbf{Q})}[x])$ -module

$$\tilde{\mathcal{N}}_{(\varphi, \beta, \mathbf{h})} = \left(\bigotimes_{l=1}^{m-1} \bigotimes_{k=1}^{n_l} L(\tilde{\omega}_l)^{\tilde{\mathbf{ev}}_{\gamma_{l,k}}} \right) \otimes \tilde{\mathcal{L}}^\beta \otimes \tilde{\mathcal{L}}^{\mathbf{h}},$$

where n_l and $\gamma_{l,k}$ ($1 \leq k \leq n_l$) are determined by $\varphi_l = (x - \gamma_{l,1})(x - \gamma_{l,2}) \dots (x - \gamma_{l,n_l})$ for each $l = 1, 2, \dots, m-1$, and we put $\beta = (\beta_i)_{1 \leq i \leq m-1}$ and $\mathbf{h} = (h_t)_{t \geq 0}$. Let $v_0^{(l,k)} \in L(\tilde{\omega}_l)^{\tilde{\mathbf{ev}}_{\gamma_{l,k}}}$ ($1 \leq l \leq m-1, 1 \leq k \leq n_l$) be a highest weight vector, $\tilde{\mathcal{L}}^\beta = \mathbb{C}w_0$ and $\tilde{\mathcal{L}}^{\mathbf{h}} = \mathbb{C}z_0$. Put $v_{(\varphi, \beta, \mathbf{h})} = (\bigotimes_{l=1}^{m-1} \bigotimes_{k=1}^{n_l} v_0^{(l,k)}) \otimes w_0 \otimes z_0 \in \tilde{\mathcal{N}}_{(\varphi, \beta, \mathbf{h})}$, then we have

$$(7.3.4) \quad \mathcal{X}_{i,t}^+ \cdot v_{(\varphi, \beta, \mathbf{h})} = 0, \quad \mathcal{I}_{j,t} \cdot v_{(\varphi, \beta, \mathbf{h})} = \tilde{\mathbf{u}}^{(\mathbf{Q})}(\varphi, \beta, \mathbf{h})_{j,t} v_{(\varphi, \beta, \mathbf{h})}$$

for $1 \leq i \leq m-1, 1 \leq j \leq m$ and $t \geq 0$ by (7.3.3). Let $\tilde{\mathcal{N}}'_{(\varphi, \beta, \mathbf{h})}$ be the $U(\mathfrak{gl}_m^{(\mathbf{Q})}[x])$ -submodule of $\tilde{\mathcal{N}}_{(\varphi, \beta, \mathbf{h})}$ generated by $v_{(\varphi, \beta, \mathbf{h})}$. Then (7.3.4) implies that $\tilde{\mathcal{N}}'_{(\varphi, \beta, \mathbf{h})}$ is a finite dimensional highest weight module of highest weight $\tilde{\mathbf{u}}^{(\mathbf{Q})}(\varphi, \beta, \mathbf{h})$. Now we obtain the following classification of finite dimensional simple $U(\mathfrak{gl}_m^{(\mathbf{Q})}[x])$ -modules.

Theorem 7.4. *For $(\varphi, \beta, \mathbf{h}) \in \prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^{(Q_i)} \times \mathbb{B}^{(Q_i)}) \times \prod_{t \geq 0} \mathbb{C}$, the highest weight simple $U(\mathfrak{gl}_m^{(\mathbf{Q})}[x])$ -module $\mathcal{L}(\tilde{\mathbf{u}}^{(\mathbf{Q})}(\varphi, \beta, \mathbf{h}))$ of highest weight $\tilde{\mathbf{u}}^{(\mathbf{Q})}(\varphi, \beta, \mathbf{h})$ is finite dimensional, and we have that*

$$\mathcal{L}(\tilde{\mathbf{u}}^{(\mathbf{Q})}(\varphi, \beta, \mathbf{h})) \cong \mathcal{L}(\tilde{\mathbf{u}}^{(\mathbf{Q})}(\varphi', \beta', \mathbf{h}')) \Leftrightarrow (\varphi, \beta, \mathbf{h}) = (\varphi', \beta', \mathbf{h}')$$

for $(\varphi, \beta, \mathbf{h}), (\varphi', \beta', \mathbf{h}') \in \prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^{(Q_i)} \times \mathbb{B}^{(Q_i)}) \times \prod_{t \geq 0} \mathbb{C}$. Moreover,

$$\{\mathcal{L}(\tilde{\mathbf{u}}^{(\mathbf{Q})}(\varphi, \beta, \mathbf{h})) \mid (\varphi, \beta, \mathbf{h}) \in \prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^{(Q_i)} \times \mathbb{B}^{(Q_i)}) \times \prod_{t \geq 0} \mathbb{C}\}$$

gives a complete set of isomorphism classes of finite dimensional simple $U(\mathfrak{gl}_m^{(\mathbf{Q})}[x])$ -modules.

We also have the following corollary.

Corollary 7.5. *For $(\varphi, \beta, \mathbf{h}) \in \prod_{i=1}^{m-1} (\mathbb{C}[x]_{\text{monic}}^{(Q_i)} \times \mathbb{B}^{(Q_i)}) \times \prod_{t \geq 0} \mathbb{C}$, we have*

$$\mathcal{L}(\tilde{\mathbf{u}}^{(Q)}(\varphi, \beta, \mathbf{h})) \cong \mathcal{L}(\mathbf{u}^{(Q)}(\varphi, \beta)) \text{ as } U(\mathfrak{sl}_m^{(Q)}[x])\text{-modules}$$

when we restrict the action on $\mathcal{L}(\tilde{\mathbf{u}}^{(Q)}(\varphi, \beta, \mathbf{h}))$ to $U(\mathfrak{sl}_m^{(Q)}[x])$.

Proof. We prove that $\mathcal{L}(\tilde{\mathbf{u}}^{(Q)}(\varphi, \beta, \mathbf{h}))$ is also simple when we restrict the action to $U(\mathfrak{sl}_m^{(Q)}[x])$. Then the isomorphism follows from the definitions of $\tilde{\mathbf{u}}^{(Q)}(\varphi, \beta, \mathbf{h})$ and $\mathbf{u}^{(Q)}(\varphi, \beta)$.

Let $v_0 \in \mathcal{L}(\tilde{\mathbf{u}}^{(Q)}(\varphi, \beta, \mathbf{h}))$ be a highest weight vector as the $U(\mathfrak{gl}_m^{(Q)}[x])$ -module. Then we have

$$\mathcal{L}(\tilde{\mathbf{u}}^{(Q)}(\varphi, \beta, \mathbf{h})) = U(\mathfrak{n}^-) \cdot v_0$$

by the triangular decomposition in Proposition 1.4 (iv). This implies that

$$(7.5.1) \quad \mathcal{L}(\tilde{\mathbf{u}}^{(Q)}(\varphi, \beta, \mathbf{h})) = U(\mathfrak{sl}_m^{(Q)}[x]) \cdot v_0.$$

Assume that $\mathcal{L}(\tilde{\mathbf{u}}^{(Q)}(\varphi, \beta, \mathbf{h}))$ is not simple as a $U(\mathfrak{sl}_m^{(Q)}[x])$ -module by the restriction, then $\mathcal{L}(\tilde{\mathbf{u}}^{(Q)}(\varphi, \beta, \mathbf{h}))$ contains a non-zero proper simple $U(\mathfrak{sl}_m^{(Q)}[x])$ -submodule which is a highest weight $U(\mathfrak{sl}_m^{(Q)}[x])$ -module. This implies that there exist an element $w_0 \in \mathcal{L}(\tilde{\mathbf{u}}^{(Q)}(\varphi, \beta, \mathbf{h}))$ such that $\mathcal{X}_{i,t}^+ \cdot w_0 = 0$ ($1 \leq i \leq m-1$, $t \geq 0$) and $w_0 \notin \mathbb{C}v_0$. Then $U(\mathfrak{gl}_m^{(Q)}[x]) \cdot w_0$ turns out to be a non-zero proper $U(\mathfrak{gl}_m^{(Q)}[x])$ -submodule of $\mathcal{L}(\tilde{\mathbf{u}}^{(Q)}(\varphi, \beta, \mathbf{h}))$. This is a contradiction. \square

APPENDIX A. SOME COMBINATORICS

A.1. Let $\mathbb{Z}[x_1, \dots, x_n]$ be the ring of polynomials in independent variables x_1, \dots, x_n over \mathbb{Z} . For $k \in \mathbb{Z}_{>0}$, put

$$\begin{aligned} p_k(x_1, \dots, x_n) &= x_1^k + x_2^k + \dots + x_n^k \in \mathbb{Z}[x_1, \dots, x_n], \\ e_k(x_1, \dots, x_n) &= \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} x_{i_1} x_{i_2} \dots x_{i_k} \in \mathbb{Z}[x_1, \dots, x_n]. \end{aligned}$$

Namely, $p_k(x_1, \dots, x_n)$ is the power sum symmetric polynomial of degree k , and $e_k(x_1, \dots, x_n)$ is the elementary symmetric polynomial of degree k . We also put $e_0(x_1, \dots, x_n) = 1$. Then, for $k > 0$, we have

$$(A.1.1) \quad ke_k(x_1, \dots, x_n) = \sum_{z=1}^k (-1)^{z-1} p_z(x_1, \dots, x_n) e_{k-z}(x_1, \dots, x_n)$$

by [M, §1 (2.11')]. For $s > n$, we have

$$0 = \sum_{z=1}^s (-1)^{z-1} p_z(x_1, \dots, x_n) e_{s-z}(x_1, \dots, x_n)$$

$$\begin{aligned}
&= \sum_{z=1}^{s-1} (-1)^{z-1} p_z(x_1, \dots, x_n) e_{s-z}(x_1, \dots, x_n) + (-1)^{s-1} p_s(x_1, \dots, x_n) \\
&= \sum_{z=s-n}^{s-1} (-1)^{z-1} p_z(x_1, \dots, x_n) e_{s-z}(x_1, \dots, x_n) + (-1)^{s-1} p_s(x_1, \dots, x_n),
\end{aligned}$$

where we note that $e_{s-z}(x_1, \dots, x_n) = 0$ if $z < s - n$. Put $w = z - s + n$, we have

$$(A.1.2) \quad \sum_{w=0}^{n-1} (-1)^{n-w+1} p_{s-n+w}(x_1, \dots, x_n) e_{n-w}(x_1, \dots, x_n) = p_s(x_1, \dots, x_n)$$

for $s > n$.

Lemma A.2. For $n \in \mathbb{Z}_{>0}$ and $u_1, u_2, \dots, u_n \in \mathbb{C}$, the simultaneous equations

$$(A.2.1) \quad \begin{cases} p_1(x_1, x_2, \dots, x_n) = u_1, \\ p_2(x_1, x_2, \dots, x_n) = u_2, \\ \vdots \\ p_n(x_1, x_2, \dots, x_n) = u_n \end{cases}$$

has a solution in \mathbb{C} .

Proof. We prove the lemma by the induction on n . In the case where $n = 1$, it is clear. If $n > 1$, the equations (A.2.1) are equivalent to the equations

$$(A.2.2) \quad \begin{cases} p_1(x_1, x_2, \dots, x_{n-1}) = u_1 - x_n, \\ p_2(x_1, x_2, \dots, x_{n-1}) = u_2 - x_n^2, \\ \vdots \\ p_{n-1}(x_1, x_2, \dots, x_{n-1}) = u_{n-1} - x_n^{n-1}, \\ p_n(x_1, x_2, \dots, x_{n-1}) = u_n - x_n^n. \end{cases}$$

By (A.1.1), we have

$$\begin{aligned}
&p_n(x_1, x_2, \dots, x_{n-1}) \\
&= \sum_{i=1}^{n-1} (-1)^{i+n-1} p_i(x_1, x_2, \dots, x_{n-1}) e_{n-i}(x_1, x_2, \dots, x_{n-1}),
\end{aligned}$$

where we note that $e_n(x_1, x_2, \dots, x_{n-1}) = 0$. On the other hand, we can write

$$e_{n-i}(x_1, x_2, \dots, x_{n-1}) = \sum_{\lambda \vdash n-i} \alpha_\lambda p_\lambda(x_1, x_2, \dots, x_{n-1})$$

for some $\alpha_\lambda \in \mathbb{C}$, where $p_\lambda(x_1, x_2, \dots, x_{n-1}) = \prod_{j=1}^{\ell(\lambda)} p_{\lambda_j}(x_1, x_2, \dots, x_{n-1})$ for $\lambda = (\lambda_1, \lambda_2, \dots) \vdash n - i$. Thus we have

$$p_n(x_1, x_2, \dots, x_{n-1}) = \sum_{i=1}^{n-1} \sum_{\lambda \vdash n-i} (-1)^{i+n-1} \alpha_\lambda p_i(x_1, x_2, \dots, x_{n-1}) p_\lambda(x_1, x_2, \dots, x_{n-1}).$$

(Note that $\{p_\mu(x_1, x_2, \dots, x_{n-1}) \mid \mu \vdash k\}$ is not linearly independent if $k \geq n$. For an example, we have $p_{(3)}(x_1, x_2) = \frac{3}{2}p_{(2,1)}(x_1, x_2) - \frac{1}{2}p_{(1,1,1)}(x_1, x_2)$.)

Then the equations (A.2.2) are equivalent to the equations

$$(A.2.3) \quad \begin{cases} p_1(x_1, x_2, \dots, x_{n-1}) = u_1 - x_n, \\ p_2(x_1, x_2, \dots, x_{n-1}) = u_2 - x_n^2, \\ \vdots \\ p_{n-1}(x_1, x_2, \dots, x_{n-1}) = u_{n-1} - x_n^{n-1}, \\ \sum_{i=1}^{n-1} \sum_{\lambda \vdash n-i} (-1)^{i+n-1} \alpha_\lambda (u_i - x_n^i) \prod_{j=1}^{\ell(\lambda)} (u_{\lambda_j} - x_n^{\lambda_j}) = u_n - x_n^n \quad \dots (*1). \end{cases}$$

Let β_n be a solution of the equation (*1) for the variable x_n . By the assumption of the induction, the simultaneous equations

$$\begin{cases} p_1(x_1, x_2, \dots, x_{n-1}) = u_1 - \beta_n, \\ p_2(x_1, x_2, \dots, x_{n-1}) = u_2 - \beta_n^2, \\ \vdots \\ p_{n-1}(x_1, x_2, \dots, x_{n-1}) = u_{n-1} - \beta_n^{n-1} \end{cases}$$

for variables x_1, x_2, \dots, x_{n-1} has a solution. We denote it by $(x_1, x_2, \dots, x_{n-1}) = (\beta_1, \beta_2, \dots, \beta_{n-1})$. Then $(x_1, x_2, \dots, x_n) = (\beta_1, \beta_2, \dots, \beta_n)$ gives a solution of (A.2.3). \square

A.3. We consider some modifications of the formulas (A.1.1) and (A.1.2) as follows. Let $\mathbf{b} = (b_1, \dots, b_n)$ be n independent variables, and we consider the ring of polynomials $\mathbb{Z}[x_1, \dots, x_n][b_1, \dots, b_n]$. For $k \in \mathbb{Z}_{>0}$, put

$$\begin{aligned} e_k^{(\mathbf{b})}(x_1, \dots, x_n) \\ = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} (b_{i_1} + b_{i_2} + \dots + b_{i_k}) x_{i_1} x_{i_2} \dots x_{i_k} \in \mathbb{Z}[x_1, \dots, x_n][b_1, \dots, b_n] \end{aligned}$$

and

$$p_k^{(\mathbf{b})}(x_1, \dots, x_n) = b_1 x_1^k + b_2 x_2^k + \dots + b_n x_n^k \in \mathbb{Z}[x_1, \dots, x_n][b_1, \dots, b_n].$$

We also put $e_0^{(\mathbf{b})} = 1$. Note that $e_k^{(\mathbf{b})}(x_1, \dots, x_n) = 0$ if $k > n$. Put $\mathbf{1} = (1, 1, \dots, 1)$, then we have $e_k^{(\mathbf{1})}(x_1, \dots, x_n) = ke_k(x_1, \dots, x_n)$ and $p_k^{(\mathbf{1})}(x_1, \dots, x_n) = p_k(x_1, \dots, x_n)$.

We consider the generating functions $E(t)$, $E^{(\mathbf{b})}(t)$ and $P^{(\mathbf{b})}(t)$ by

$$\begin{aligned} E(t) &= \sum_{k \geq 0} e_k(x_1, \dots, x_n) t^k \in \mathbb{Z}[x_1, \dots, x_n][b_1, \dots, b_n][[t]], \\ E^{(\mathbf{b})}(t) &= \sum_{k \geq 0} e_{k+1}^{(\mathbf{b})}(x_1, \dots, x_n) t^k \in \mathbb{Z}[x_1, \dots, x_n][b_1, \dots, b_n][[t]], \\ P^{(\mathbf{b})}(t) &= \sum_{k \geq 0} (-1)^k p_{k+1}^{(\mathbf{b})}(x_1, \dots, x_n) t^k \in \mathbb{Z}[x_1, \dots, x_n][b_1, \dots, b_n][[t]]. \end{aligned}$$

Then, we have

$$E(t) = \prod_{i=1}^n (1 + x_i t), \quad P^{(\mathbf{b})}(t) = \sum_{i=1}^n \frac{b_i x_i}{1 + x_i t}$$

and

$$P^{(\mathbf{b})}(t) E(t) = \sum_{i=1}^n b_i x_i \left(\prod_{j=1, j \neq i}^n (1 + x_j t) \right) = E^{(\mathbf{b})}(t).$$

This implies that, for $k \geq 0$,

$$(A.3.1) \quad e_{k+1}^{(\mathbf{b})}(x_1, \dots, x_n) = \sum_{z=0}^k (-1)^z p_{z+1}^{(\mathbf{b})}(x_1, \dots, x_n) e_{k-z}(x_1, \dots, x_n).$$

In the case where $k = n$, we have

$$\sum_{z=0}^n (-1)^z p_{z+1}^{(\mathbf{b})}(x_1, \dots, x_n) e_{n-z}(x_1, \dots, x_n) = 0$$

since $e_{n+1}^{(\mathbf{b})}(x_1, \dots, x_n) = 0$. This implies that

$$(A.3.2) \quad \sum_{z=0}^{n-1} (-1)^{n-z+1} p_{z+1}^{(\mathbf{b})}(x_1, \dots, x_n) e_{n-z}(x_1, \dots, x_n) = p_{n+1}^{(\mathbf{b})}(x_1, \dots, x_n).$$

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, SHINSHU UNIVERSITY, ASAHI 3-1-1, MATSUMOTO 390-8621, JAPAN

E-mail address: `wada@math.shinshu-u.ac.jp`